ANALYTICAL RECONSTRUCTION FOR A WEIGHTED INTEGRAL GEOMETRY PROBLEM USING THE FOURIER TRANSFORM

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FURYE ALMAShTIRIShI YORDAMIDA VAZNLI INTEGRAL GEOMETRIYA MASALASINI ANALITIK TIKLASh

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АНАЛИТИЧЕСКАЯ РЕКОНСТРУКЦИЯ ДЛЯ ЗАДАЧИ ВЗВЕШИВАНИЯ ИНТЕГРАЛЬНОЙ ГЕОМЕТРИИ С ИСПОЛЬЗОВАНИЕМ ПРЕОБРАЗОВАНИЯ ФУРЬЕ

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Tayanch so'zlar: integral geometriya, teskari masala, vaznli integral almashtirish, polinomial vazn, Furye almashtirishi, teskari formula, yagonalik.

Rezyume. Maqolada vaznli integral ma'lumotlar asosida ikki o'zgaruvchili funksiyani tiklashga oid integral geometriyaning teskari masalasi qaraladi. Vazn funksiyasi polinomial ko'rinishda beriladi. Fazoviy o'zgaruvchilar bo'yicha Furye almashtirishini qo'llash orqali boshlang'ich integral tenglama chastotalar sohasidagi ekvivalent munosabatga keltiriladi. Hosil bo'lgan ifoda tahlil qilinib, teskari Furye almashtirishlari yordamida noma'lum funksiyani tiklashning oshkor formulasi olinadi. Yechimning yagonaligini ta'minlovchi shartlar aniqlanadi. Taklif etilgan yondashuv vaznli integral ma'lumotlar asosida funksiyalarni analitik tiklash uchun matematik asos yaratadi.

Ключевые слова: интегральная геометрия, обратная задача, взвешенное интегральное преобразование, полиномиальный вес, преобразование Фурье, формула обращения, единственность.

Резюме. В работе рассматривается обратная задача интегральной геометрии о восстановлении функции двух переменных по заданным взвешенным интегральным данным. Весовая функция задается в полиномиальном виде. С помощью преобразования Фурье по пространственным переменным исходное интегральное уравнение сводится к эквивалентному соотношению в частотной области. Полученное представление исследуется, и с использованием обратных преобразований Фурье выводится явная формула восстановления неизвестной функции. Устанавливаются условия, обеспечивающие единственность решения. Предложенный подход формирует аналитическую основу для восстановления функций по взвешенным интегральным данным.

Key words: integral geometry, inverse problem, weighted integral transform, polynomial weight, Fourier transform, inversion formula, uniqueness.

Summary. This paper considers an inverse problem of integral geometry for reconstructing a two-dimensional function from weighted integral data. The weight function is assumed to have a polynomial form. By applying the Fourier transform with respect to the spatial variables, the original integral equation is transformed into an equivalent relation in the frequency domain. The resulting representation is analyzed and an explicit formula for recovering the unknown function is obtained by means of inverse Fourier transforms. Conditions ensuring the uniqueness of the solution are established. The proposed approach provides an analytical framework for the reconstruction of functions from weighted integral data.

Introduction. Inverse problems of integral geometry constitute an important area of modern applied mathematics. They provide a mathematical framework for reconstructing unknown functions from indirect integral measurements and arise naturally in tomography, mathematical imaging, non-destructive testing, and inverse problems for differential and kinetic equations. Fundamental theoretical aspects of integral geometry and its applications to inverse problems have been systematically investigated in [1]. The development of modern imaging technologies has considerably increased the importance of stable reconstruction methods. In particular, inverse scattering techniques used in high-resolution imaging demonstrate that internal characteristics of an object can be recovered from indirect measurement data by means of appropriately constructed mathematical models [2]. A significant class of inverse problems is associated with weighted integral transforms. In such problems, the unknown function is integrated together with a prescribed weight function. The introduction of a nonconstant or discontinuous weight substantially changes the analytical properties of the corresponding integral operator and may complicate both the derivation of inversion formulas and their numerical implementation. Integral geometry problems involving nontrivial weight functions have been investigated in [3]. Another important research direction concerns integral transforms defined over special families of lines and curves. Analytical inversion procedures for the V-line Radon transform have been developed in [4]. These results show that reconstruction from integral data collected along piecewise-linear trajectories can be formulated within the general framework of integral geometry. Integral geometry problems arising in tomography and related reconstruction applications have also been extensively studied [5]. Such investigations demonstrate that the geometry of the integration family has a fundamental influence on uniqueness, stability, and the construction of explicit inversion formulas. Further applications of integral-geometric methods to advanced imaging problems have been considered in [6]. These studies emphasize the importance of developing reconstruction techniques

adapted to the specific geometry of the measurement process. Theoretical and computational aspects of the V-line Radon transform and its applications to imaging were further developed in [7]. The obtained results provide additional analytical tools for recovering unknown functions from integral measurements performed over nonstandard trajectories. A closely related class of inverse problems involves the broken-ray Radon transform. Explicit inversion formulas and reconstruction techniques for this transform have been investigated in [8]. Such models are particularly relevant when measurement trajectories consist of several connected line segments rather than conventional straight lines. Certain classes of integral geometry problems can also be related to inverse and boundary-value problems for differential equations. Classical results establishing such connections and providing approaches to uniqueness and reconstruction were obtained in [9]. A general theoretical framework for integral geometry and generalized Radon-type transforms is presented in [10]. These results form an important mathematical basis for studying integral transforms defined over different families of geometrical objects and for constructing corresponding inversion procedures. Of particular relevance to the present investigation is the reconstruction of functions from weighted integral data collected over a family of straight-line segments. An analytical reconstruction approach for a polynomial weight function was considered in [11]. This formulation provides the mathematical basis for developing stable numerical algorithms for the corresponding weighted integral equation.

Statement of the problem. Consider the following equation:

$$\int\limits_{\Gamma(x,y)} g(x,y,\xi,\eta)u(\xi,\eta)d\eta = f(x,y) \tag{1}$$

$$\Gamma(x,y)=\{(\xi,\eta):\ \ |x-\xi|=y-\eta,\ \ 0\le\eta\le y\le H\ \ \}.$$

The problem of solving an equation of this form has been considered in other scientific works, and the corresponding analytical inversion formula has been derived and $g(x,\xi)=a_0+a_1(x-\xi)\ \ a_2=0$. **Numerical experiments have been carried out for this case [11].**

Theorem. Let the function $f(x,y)$ be known for all $(x,y)\in L_H$ and let the weight function be of the form $g(x,\xi)=a_0+a_1(x-\xi)+a_2(x-\xi)^2,\ a_0,a_1,a_2\in R,\ a_0\neq 0.$

Then, in the class of functions $C_0^2(L_H)$ the solution of equation (1) is **unique** and can be represented explicitly as

$$u(x,y)=\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\frac{\sqrt{2}\tilde{\tilde{f}}(\lambda,\mu)(\lambda^2\tau^2-\mu^2)^3 e^{-i\mu y-i\lambda x}}{ia_0\mu(\lambda^2\tau^2-\mu^2)-a_1(\lambda^2\tau^2-\mu^2)^2-2ia_2\mu(\mu^2+3\lambda^2\tau^2)}d\mu d\lambda$$

Proof. We rewrite equation (1) in the following equivalent integral form:

$$\frac{1}{\sqrt{2}}\int\limits_0^y\left(a_0+a_1(y-\eta)+a_2(y-\eta)^2\right)u\left(x-(y-\eta),\eta\right)d\eta+$$

$$\frac{1}{\sqrt{2}}\int\limits_0^y\left(a_0+a_1(y-\eta)+a_2(y-\eta)^2\right)u\left(x+(y-\eta),\eta\right)d\eta=f(x,y) \tag{2}$$

Next, we apply the Fourier transform with respect to the spatial variable x to both sides of equation (2):

$$\int\limits_0^y\hat{u}(\lambda,\eta)\left(a_0+a_1(y-\eta)+a_2(y-\eta)^2\right)\cos\left(\lambda(y-\eta)\right)d\eta=\sqrt{2}\hat{f}(\lambda,y) \tag{3}$$

We now apply the Fourier transform with respect to the variable y to both sides of the last equation

$$\int\limits_0^{+\infty}e^{i\mu y}\int\limits_0^y\hat{u}(\lambda,\eta)(a_0+a_1(y-\eta)+a_2(y-\eta)^2)\cos(\lambda(y-\eta))d\eta dy=$$

$$=\int\limits_0^{+\infty}\int\limits_0^{+\infty}e^{i\mu\eta}e^{i\mu\tau}\hat{u}(\lambda,\eta)\left(a_0+a_1\tau+a_2\tau^2\right)\cos(\lambda\tau)d\tau d\eta=$$

$$=\int\limits_0^{+\infty}e^{i\mu\eta}u(\lambda,\eta)d\eta\int\limits_0^{+\infty}e^{-\mu\tau}\left(a_0+a_1\tau+a_2\tau^2\right)\cos(\lambda\tau)d\tau$$

$$\tilde{\hat{u}}(\lambda,\mu)I(\lambda,\mu)=\tilde{\tilde{f}}(\lambda,\mu)$$

or

$$\tilde{\hat{u}}(\lambda,\mu)=\frac{\tilde{\tilde{f}}(\lambda,\mu)}{I(\lambda,\mu)}$$

where $I(\lambda,\mu)=\int\limits_0^{+\infty}e^{-i\mu\tau}\left(a_0+a_1\tau+a_2\tau^2\right)\cos(\lambda\tau)d\tau,$

$\tilde{\hat{u}}(\lambda,\mu)=\int\limits_0^{+\infty}e^{i\mu\eta}u(\lambda,\eta)d\eta,\ \ \tilde{\hat{f}}(\lambda,y)=\int\limits_0^{+\infty}e^{i\mu y}\hat{f}(\lambda,y)dy.$

As a result, we obtain the following explicit expression:

$$\tilde{\hat{u}}(\lambda,\mu)=\frac{\sqrt{2}\tilde{\tilde{f}}(\lambda,\mu)}{\left(\frac{ia_0\mu}{\lambda^2\tau^2-\mu^2}-a_1\frac{\mu^2+\lambda^2\tau^2}{(\lambda^2\tau^2-\mu^2)^2}+a_2\frac{2i\mu(-\mu^2-3\lambda^2\tau^2)}{(\lambda^2\tau^2-\mu^2)^3}\right)}$$

Applying the inverse Fourier transforms with respect to the variables μ and λ, we obtain the desired solution.

The numerical evaluation of the obtained integral representation is associated with substantial difficulties arising from both the analytical structure of the kernel and the properties of the Fourier transforms involved. First, the denominator of the integral contains singular factors of the form $(\lambda^2\tau^2-\mu^2)^{-k}$, which lead to the appearance of poles and cause the integrand to grow sharply in the vicinity of critical points. This significantly increases sensitivity to noise in the data and amplifies the inherent ill-posedness of the problem. These factors make the direct evaluation of the integral impractical without employing regularization and spectral filtering methods that ensure numerical stability and physical reliability of the reconstructed solution.

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