ShVARS-KRISTOFFEL FORMULASI VA UNING SOHALARNI KONFORM AKSLANTIRIShDAGI TATBIQLARI

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ФОРМУЛА ШВАРЦА-КРИСТОФФЕЛЯ И ЕЁ ПРИЛОЖЕНИЯ В КОНФОРМНОМ ОТОБРАЖЕНИИ ОБЛАСТЕЙ

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SCHWARZ–CHRISTOFFEL FORMULA AND ITS APPLICATIONS IN CONFORMAL MAPPING OF DOMAINS

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Tayanch so'zlar: konform akslantirish, Shvars-Kristoffel integrali, yuqori yarim tekislik, birlik doira, kompleks o'zgaruvchili funksiyalar.

Rezyume. Konform akslantirishlar nazariyasi kompleks analizning eng amaliy bo'limlaridan biri bo'lib, gidrodinamika, elektrostatika va issiqlik o'tkazuvchanlik masalalarini yechishda asosiy vosita hisoblanadi. Riman teoremasiga ko'ra, har qanday bir bog'lamli sohani birlik doiraga akslantirish mumkin. Biroq, amaliy masalalarda soha chegaralari ko'pincha siniq chiziqlardan (ko'pburchaklardan) iborat bo'ladi. Bunday hollarda Shvars-Kristoffel integrali universal yechim vazifasini o'taydi. Mazkur maqolada Shvars-Kristoffel formulasining nazariy asoslari va uning tatbiqlari metodik jihatdan tahlil etilgan.

Ключевые слова: конформное отображение, интеграл Шварца-Кристоффеля, верхняя полуплоскость, единичный круг, функции комплексной переменной.

Резюме. Теория конформных отображений является одним из практических разделов комплексного анализа и служит основным инструментом при решении задач гидродинамики, электростатики и теплопроводности. В сложных задачах, где границы области состоят из многоугольников, формула Шварца-Кристоффеля выступает универсальным решением. В данной статье анализируются теоретические основы формулы Шварца-Кристоффеля и её методическое применение.

Key words: conformal mapping, Schwarz-Christoffel integral, upper half-plane, unit disk, functions of a complex v ariable.

Summary. The theory of conformal mapping is one of the most practical branches of complex analysis and serves as the main tool for solving problems in hydrodynamics, electrostatics, and heat transfer. In complex problems where domain boundaries consist of broken lines (polygons), the Schwarz-Christoffel formula serves as a universal solution. This paper analyzes the theoretical and methodological foundations of the Schwarz-Christoffel formula, as well as problems of mapping the upper half-plane onto the interior of polygons.

Kirish. Konform akslantirishlar nazariyasi kompleks analizning eng amaliy bo'limlaridan biri bo'lib, gidrodinamika, elektrostatika va issiqlik o'tkazuvchanlik masalalarini yechishda asosiy vosita hisoblanadi. Riman teoremasiga ko'ra, har qanday bir bog'lamli sohani birlik doiraga akslantirish mumkin. Biroq, amaliy masalalarda soha chegaralari ko'pincha siniq chiziqlardan (ko'pburchaklardan) iborat bo'ladi. Bunday hollarda Shvars-Kristoffel integrali universal yechim vazifasini o'taydi.

Metodlar va nazariy asoslar.

Shvars-Kristoffel formulasining nazariy asoslari: Aytaylik, w-tekisligida n ta uchli $A_1, A_2, \dots, A_n$ ko'pburchak berilgan bo'lsin va uning ichki burchaklari $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$ ga teng bo'lsin. Yuqori yarim tekislikni ($\operatorname{Im} z > 0$) ushbu ko'pburchak ichiga konform akslantiruvchi funksiya quyidagi ko'rinishdagi integral orqali ifodalanadi:

$$w = f(z) = C\int_{z_0}^{z} (\zeta - x_1)^{\alpha_1 - 1}(\zeta - x_2)^{\alpha_2 - 1}\dots(\zeta - x_n)^{\alpha_n - 1} d\zeta + C_1$$

bunda $x_1 < x_2 < \dots < x_n$ - haqiqiy o'qdagi nuqtalar bo'lib, ular w-tekisligidagi ko'pburchak uchlarining proobrazlaridir, C va C_1 kompleks o'zgarmaslar bo'lib, sohaning o'lchami va tekislikdagi o'rnini belgilaydi, $\alpha_k - 1$ daraja ko'rsatkichi, bunda $\alpha_k\pi$ ko'pburchakning k-uchidagi ichki burchagi.

Ko'pburchak uchlaridan biri cheksiz uzoqlashgan nuqtaning obrazi bo'lgan holat: Faraz qilaylik, $a_n = \infty$ bo'lsin. Bu holatni avval ko'rilgan holatga keltirish uchun, $\operatorname{Im} z > 0$ yarim tekisligini $\operatorname{Im}\zeta > 0$ yarim tekisligiga o'tkazuvchi va $a_1, a_2, \dots, a_n = \infty$ nuqtalarni chekli $a_{1'}, a_{2'}, \dots, a_{n'}$ nuqtalarga o'tkazuvchi $\zeta = -\frac{1}{z} + a_{n'}$

chiziqli almashtirishni bajaramiz. Formulani qo‘llab, quyidagiga ega bo‘lamiz:

$$w=C'\int_{\zeta_0}^{\zeta}(\zeta-a_{1'})^{\alpha_1-1}(\zeta-a_{2'})^{\alpha_2-1}\dots(\zeta-a_{n'})^{\alpha_n-1}d\zeta+C_1=$$

$$=C'\int_{z_0}^{z}\left(a_{n'}-a_{1'}-\frac{1}{z}\right)^{\alpha_1-1}\left(a_{n'}-a_{2'}-\frac{1}{z}\right)^{\alpha_2-1}\dots\left(-\frac{1}{z}\right)\frac{dz}{z^2}+C_1\cdot$$

Har bir qavs ichidagi ifodani umumiy maxrajga keltiramiz va har bir qavsdan $(a_{n'}-a_{k'})$ ($k=1,\dots,n-1$) ko‘paytuvchilarni qavsdan tashqariga chiqaramiz; u holda quyidagiga ega bo‘lamiz:

$$w=C\int_{z_0}^{z}(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}\dots$$

$$\dots(z-a_{n-1})^{\alpha_{n-1}-1}\frac{dz}{z^{\alpha_1+\alpha_2+\dots+\alpha_n-n+2}}+C_1$$

bu yerda $a_k=\dfrac{1}{a_{n'}-a_{k'}}$ — ba'zi haqiqiy o‘zgarmaslar, C esa kompleks o‘zgarmasdir. n-burchakning ichki burchaklari yig‘indisi haqidagi elementar geometrik teoremaga ko‘ra: $\alpha_1+\alpha_2+\dots+\alpha_n=n-2$ ekanligidan foydalanib, yakuniy formulani hosil qilamiz:

$$w=C\int_{z_0}^{z}(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}\dots(z-a_{n-1})^{\alpha_{n-1}-1}dz+C_1$$

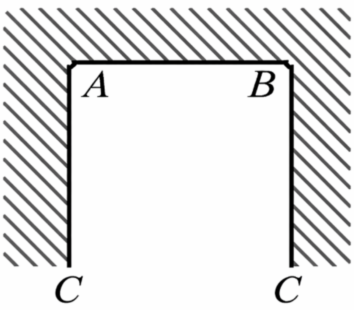
Birlik doira uchun Shvars-Kristoffel formulasi: $w_1,w_2,\dots,w_n$ nuqtalar Δ ko‘pburchakning uchlari, $\alpha_1\pi,\alpha_2\pi,\dots,\alpha_n\pi$ esa ushbu uchlarga mos keluvchi ichki burchaklar bo‘lsin. Agar $w=f(z)$ funksiyasi birlik doiraning ichki qismini Δ ko‘pburchakning ichki qismiga konform akslantirsa, u holda quyidagi ko‘rinishga ega bo‘ladi:

$$f(z)=C\int_{z_0}^{z}(\zeta-a_1)^{\alpha_1-1}(\zeta-a_2)^{\alpha_2-1}\dots(\zeta-a_n)^{\alpha_n-1}d\zeta+C_1$$

bu yerda z_0,C,C_1 va $|a_k|=1$ ($k=1,2,\dots,n$) ba’zi o‘zgarmaslardir.

Natijalar (Misollar tahlili)

1-Misol: $\operatorname{Im} z>0$ yuqori yarim tekislikni 1-rasmda ko‘rsatilgan ω-tekisligidagi sohaga akslantiring. Nuqtalar mosligi: $\omega(A=0,B=1,C=\infty)\leftarrow z(0,1,\infty)$ deb berilgan.



1-rasm

Yechish: 1-rasmdan ko‘rinib turibdiki, ushbu uchburchakning burchaklari mos ravishda $\alpha_A=\frac{3}{2},\alpha_B=\frac{3}{2}$ ga teng, bundan $\alpha_C=-2$ kelib chiqadi. Shvars-Kristoffel formulasiga ko‘ra:

$$\omega(z)=C\int_0^z(\xi)^{\frac{3}{2}-1}(\xi-1)^{\frac{3}{2}-1}d\xi=C\int_0^z\sqrt{\xi}\sqrt{\xi-1}d\xi$$

.

Integrallash natijasida quyidagiga ega bo‘lamiz:

$$\omega(z)=C(\arcsin\sqrt{z}-\sqrt{z(1-z)}(1-2z)).$$

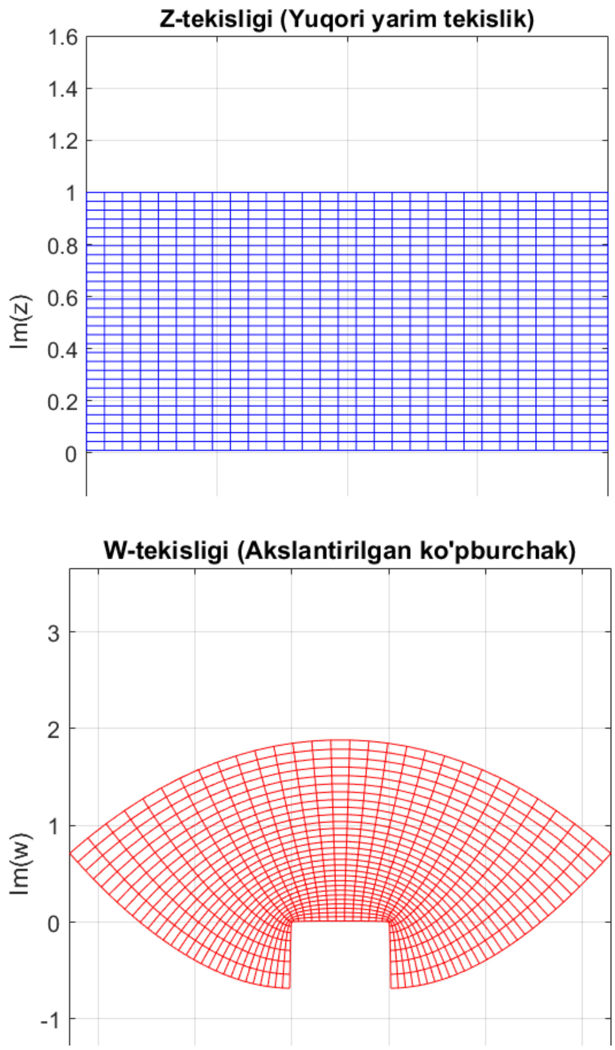
Bizda $\omega(1)=1$ sharti bor, shundan foydalanib o‘zgarmas C ni topamiz:

$$1=C(\arcsin\sqrt{1})=C\frac{\pi}{2}\Rightarrow C=\frac{2}{\pi}.$$

Demak, izlanayotgan akslantirish funksiyasi:

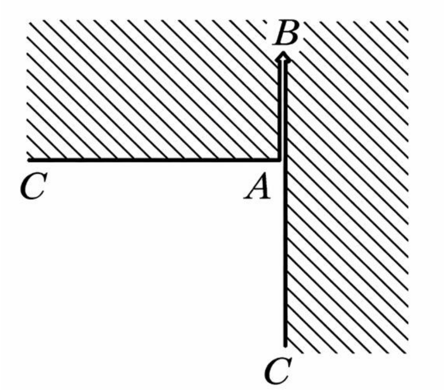
$$\omega(z)=\frac{2}{\pi}(\arcsin\sqrt{z}-\sqrt{z(1-z)}(1-2z)).$$

Demak yechimni grafik shaklida quyidagicha bo‘ladi:



2-rasm

2-Misol: $\operatorname{Im} z>0$ yuqori yarim tekislikni 3-rasmda ko‘rsatilgan ω-tekisligidagi sohaga akslantiring. Nuqtalar mosligi: $\omega(A=0,B=ia,C=\infty)\leftarrow z(0,1,\infty)$ deb berilgan.



3-rasm

Yechish: 3-rasmdan ko‘rinib turibdiki, burchaklar $\alpha_A = \frac{1}{2}, \alpha_B = 2$ ga teng, bundan $\alpha_C = -\frac{3}{2}$ kelib chiqadi. Shvars-Kristoffel integraliga ko‘ra:

$$f(z) = C\int_0^z t^{\frac{1}{2}-1}(t-1)^{2-1}dt = C\int_0^z \frac{t-1}{\sqrt{t}}dt = C\left(\frac{2}{3}z^{3/2} - 2\sqrt{z}\right)$$

Endi o‘zgarmas C ni topamiz:

$$f(1) = C\left(\frac{2}{3} - 2\right) = -\frac{4}{3}C = ia \Rightarrow C = -\frac{3ai}{4}.$$

Demak, yakuniy funksiya:

$$f(z) = -\frac{3ai}{4}\left(\frac{2}{3}z^{3/2} - 2\sqrt{z}\right) = -\frac{ai}{2}(z^{3/2} - 3\sqrt{z}).$$

3-Misol: $|z|<1$ doirani koordinata boshida markazlashgan va uchlaridan biri $w=1$ nuqtada bo‘lgan muntazam besh burchakli yulduz ichiga konform akslantiruvchi $w(z)$ funksiyasini toping. Bunda $w(0)=0$ va $w'(0)>0$ shartlari bajarilsin.

Yechish: Faraz qilaylik, $\omega(1)=1$ bo‘lsin. Ushbu besh burchakli yulduzda beshta o‘tkir burchak o‘zaro teng va yana beshta botiq (o‘tmas) burchak ham o‘zaro tengdir. Ko‘pburchak ichki burchaklari yig‘indisi haqidagi teoremaga ko‘ra $5\alpha + 5\beta = 8$ munosabat o‘rinli, bunda $\alpha\pi$ — o‘tkir burchak, $\beta\pi$ — botiq burchak ko‘rsatkichi. Shvars-Kristoffel formulasini quyidagicha yozish mumkin:

$$\omega(z) = C\int_0^z \prod_{k=1}^n (a_k - \xi)^{\alpha_k - 1} d\xi =$$

$$= C\int_0^z \prod_{k=1}^5 \left(e^{i\frac{2\pi(k-1)}{5}} - \xi\right)^{\alpha-1} \prod_{j=1}^5 \left(e^{i\frac{2\pi(j-1)}{5}+\frac{\pi}{5}} - \xi\right)^{\beta-1} d\xi =$$

$$= C\int_0^z (1-\xi^5)^{\alpha-1}(1+\xi^5)^{\beta-1} d\xi.$$

Bizga ma'lumki, ushbu yulduzning ichki burchaklari mos ravishda $\alpha\pi = \frac{1}{5}\pi$ va $\beta\pi = \frac{7}{5}\pi$ ga teng. Bundan quyidagi ifoda kelib chiqadi:

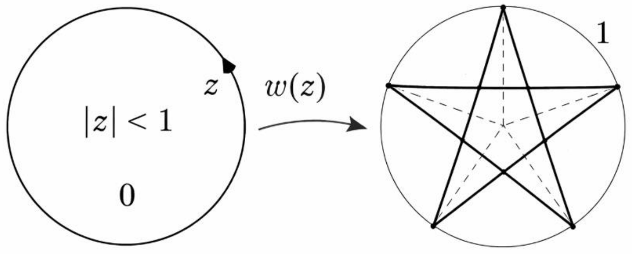
$$\omega(z) = C\int_0^z (1-\xi^5)^{-\frac{4}{5}}(1+\xi^5)^{\frac{2}{5}} d\xi$$

Endi C o‘zgarmasini aniqlaymiz. $\omega(1)=1$ shartidan foydalansak:

$$C\int_0^1 (1-\xi^5)^{-\frac{4}{5}}(1+\xi^5)^{\frac{2}{5}} d\xi = 1$$

bu yerda integral Beta-funksiya orqali hisoblanib, natijada C quyidagiga teng bo‘ladi:

$$C = \frac{5\cdot 2^{2/5}}{B(1/10, 1/5)}.$$



4-rasm

Muhokama: Shvars-Kristoffel formulasi muhandislik masalalarini matematik modellashtirishda kuchli vositadir. Metodik jihatdan ushbu mavzu talabalarda integral hisobini geometrik interpretatsiya qilish, kompleks o‘zgaruvchili funksiyalar geometriyasini chuqur anglash ko‘nikmalarini shakllantiradi. Maqolada keltirilgan nazariy hisob-kitoblar talabalarga konform akslantirishlarni mustaqil tahlil qilish imkonini beradi.

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