# MATRITSAVIY FAZOLARDA GARMONIK FUNKSIYALAR

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# ГАРМОНИЧЕСКИЕ ФУНКЦИИ В МАТРИЧНЫХ ПРОСТРАНСТВАХ

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# HARMONIC FUNCTIONS IN MATRIX SPACES

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**Tayanch so'zlar:** matritsaviy analiz, differensial operator, Laplas operatori, invariantlik, matritsaviy soha, garmonik funksiya.

**Rezyume.** Ushbu maqolada matritsaviy analizda kiritilgan bazi differensial operatorlarning nazariy asoslari tadqiq etiladi. Matritsa argumentli funksiyalar uchun differensial operatorning aniqlanishi, uning matritsaviy ko'rinishi hamda chiziqli kasrli almashtirishlarga nisbatan invariantlik xossalari o'rganiladi. Shuningdek, matritsaviy sohalarda aniqlangan Laplas operatorining umumlashtirilgan ko'rinishi va uning garmonik funksiyalar nazariyasidagi ahamiyati yoritiladi.

**Ключевые слова:** матричный анализ, дифференциальный оператор, оператор Лапласа, инвариантность, матричная область, гармоническая функция.

**Резюме.** В данной статье исследуются дифференциальные операторы в рамках матричного анализа. Рассматривается определение дифференциального оператора для функций матричного аргумента, его матричная форма, а также инвариантные свойства относительно линейно-дробных преобразований. Особое внимание уделяется обобщённому оператору Лапласа на матричных областях и его роли в теории гармонических функций.

**Key words:** matrix analysis, differential operator, Laplace operator, invariance, matrix domain, harmonic function.

**Summary.** This paper investigates differential operators within the framework of matrix analysis. The definition of differential operators for matrix-valued functions, their matrix representation, and their invariant properties under linear fractional transformations are studied. Particular attention is paid to the generalized Laplace operator on matrix domains and its significance in the theory of harmonic functions.

**Kirish.** Differensial operatorlar matritsaviy analizning muhim tushunchalaridan biri bo'lib, matritsa argumentli funksiyalarni o'rganishda asosiy vosita hisoblanadi. Ko'p o'lchamli matematik modellarni tahlil qilishda differensiallash jarayonini matritsa o'zgaruvchilarga nisbatan aniqlash zarurati yuzaga keladi. Ushbu maqolada matritsaviy analiz doirasida kiritilgan differensial operatorlarning tuzilishi, ularning matritsaviy ko'rinishi hamda chiziqli kasrli almashtirishlarga nisbatan invariantlik xossalari tadqiq etiladi. Shuningdek, matritsaviy sohalarda aniqlangan Laplas operatorining umumlashtirilgan shakli va uning garmonik funksiyalar nazariyasidagi o'rni yoritiladi.

**Asosiy qism.** Ushbu

$$\Re_I = \left\{ Z \in \mathbb{C}[m\times n]:\ I^{(m)} - ZZ^* > 0 \right\},$$

soha, (Kartan tasnifi bo'yicha) birinchi tip klassik soha deb nomlanadi. Bu yerda $I^{(m)}$– birlik $[m\times m]$- tartibli matritsa, $Z^*$– matritsa esa transponirlangan $Z'$ matritsaning kompleks qo'shmasi, $I^{(m)} - ZZ^* > 0$ tengsizlik $I^{(m)} - ZZ^*$ - ifoda Ermit matritsasining musbat aniqlanganligini ya'ni barcha xos sonlarining musbat ekanligini anglatadi [3:9].

Birinchi tip klassik $\Re_I$ soha $mn$ - kompleks o'lchovli, to'la doiraviy, chegaralangan va qavariq soha hisoblanadi [3,4].

$\Re_I$ sohaning ostovi (Shilov chegarasi) $G_I$ to'plam $UU^* = I^{(m)}$ shartni qanoatlantiruvchi $U - [m\times n]$ - tartibli matritsalardan iborat. Xususiy holda, $m=n$ bo'lsa $G_I$ ostov - $[m\times m]$ tartibli unitar matritsalar to'plami bilan ustma-ust tushadi. $G_I$ ostov $m(2n-m)$ haqiqiy o'lchovga ega va bir jinsli soha hisoblanadi, ya'ni $G_I$ to'plamning har bir nuqtasini, $\Re_I$ sohadagi boshlang'ich nuqtani o'zgartirmaydigan akslantirishlar yordamida, $G_I$ to'plamga tegishli boshqa nuqtalarga o'tkazish mumkin.

Agar $m=n$ bo'lsa, $\Re_I$ - soha $\mathbb{C}[m\times m]$ fazodagi matritsaviy doira deyiladi.

*Sp* – bu kvadrat matritsaning asosiy diagonali elementlarining yig'indisi.

$P(Z,U)$ - Puasson yadrosi bo'lib klassik sohalar geometriyasida garmonik analizni va invariant metrikalarni ifodalovchi asosiy yadro funksiyasidir.

Birinchi navbatda biz Xua Lo-Kenn differensial operatorini keltiramiz [3:116].

$$\partial_z = \begin{pmatrix} \dfrac{\partial}{\partial z_{11}} \cdots \dfrac{\partial}{\partial z_{1n}} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial}{\partial z_{m1}} \cdots \dfrac{\partial}{\partial z_{mn}} \end{pmatrix}. \qquad (1)$$

Ushbu $\Gamma_1$ - sistema $\mathfrak{R}_1$ sohaning akslantirishlar guruhi bo‘lsin ([1]). Bunda $\Gamma_1$- sistema quyidagi akslantirishlar-dan iborat:

$$W=(AZ+B)(CZ+D)^{-1}=(ZB^*+A^*)^{-1}(ZD^*+C^*). \qquad (2)$$

Bunda A - $m\times m$ matritsa, B - $m\times n$ matritsa, C - $n\times m$ matritsa, D - $n\times n$ matritsa bo‘ladi. Bu matritsalar

$$AA^*-BB^*=I,\ AC^*=BD^*,\ CC^*-DD^*=-I, \qquad (3)$$

shartlarini qanotlandiradi, ularni

$$A^*A-C^*C=I,\ \ A^*B=C^*D,\ \ B^*B-D^*D=-I \qquad (4)$$

ko‘rinishida ham yozish mumkin.

(2) ko‘rinishidagi akslantirishlarning o‘z-ora tengligi (3) va (4) nisbatlardan quyidagicha isbotlanadi:

$$(ZB^*+A^*)(AZ+B)=(ZD^*+C^*)(CZ+D)\Leftrightarrow$$
$$\Leftrightarrow ZB^*AZ+ZB^*B+A^*AZ+A^*B=ZD^*CZ+ZD^*D+C^*CZ+C^*D\Leftrightarrow$$
$$\Leftrightarrow ZB^*AZ-ZD^*CZ+ZB^*B-ZD^*D+A^*B-C^*D=0\Leftrightarrow$$
$$\Leftrightarrow Z(B^*A-D^*C)Z+Z(B^*B-D^*D)=0$$

Endi (2) ni differensiallasak, u holda (3) shartga ko‘ra quyidagiga ega bo‘lamiz:

$$dW=\left[A(CZ+D)^{-1}-(AZ+B)(CZ+D)^{-2}C)\right]dZ=$$
$$=\left[A-(AZ+B)(CZ+D)^{-1}C\right]dZ(CZ+D)^{-1}=$$
$$=\left[A-(ZB^*+A^*)^{-1}(ZD^*+C^*)C\right]dZ(CZ+D)^{-1}=$$
$$=(ZB^*+A^*)^{-1}\left[(ZB^*+A^*)A-ZD^*C-C^*C\right]dZ(CZ+D)^{-1}=$$
$$=(ZB^*+A^*)^{-1}(ZB^*A+A^*A-ZD^*C-C^*C)dZ(CZ+D)^{-1}=$$
$$=(ZB^*+A^*)^{-1}dZ(CZ+D)^{-1}.$$

Agar akslantirishni

$$W_1=(AZ_1+B)(CZ_1+D)^{-1}$$

ko‘rinishda belgilasak, u holda quyidagi

$$I-W_1^*W=I-\left(Z_1^*C^*+D^*\right)^{-1}\left(Z_1^*A^*+B^*\right)$$
$$\left(AZ+B\right)\left(CZ+D\right)^{-1}=$$
$$=\left(Z_1^*C^*+D^*\right)^{-1}\left[\left(Z_1^*C^*+D^*\right)\left(CZ+D\right)-\right.$$
$$\left.-\left(Z_1^*A^*+B\right)^*\left(AZ+B\right)\right]\left(CZ+D\right)^{-1}=$$
$$=\left(Z_1^*C^*+D^*\right)^{-1}\left[Z_1^*\left(C^*C-A^*A\right)Z+Z_1^*\left(C^*D-A^*B\right)+\right.$$
$$\left.+\left(D^*C-B^*A\right)Z+D^*D-B^*B\right]\left(CZ+D\right)^{-1}=$$
$$=\left(Z_1^*C^*+D^*\right)^{-1}\left(I-Z_1^*Z\right)\left(CZ+D\right)^{-1}. \qquad (5)$$

yoki

$$I-WW_1^*=I-(ZB^*+A^*)^{-1}(ZD^*+C^*)(DZ_1^*+C)(BZ_1^*+A)^{-1}=$$
$$=(ZB^*+A^*)^{-1}\left[(ZB^*+A^*)(BZ_1^*+A)-\right.$$
$$\left.-(ZD^*+C^*)(DZ_1^*+C)\right](BZ_1^*+A^*)^{-1}=$$
$$=(ZB^*+A^*)^{-1}\left[ZB^*BZ_1^*+ZB^*A+A^*BZ_1^*+A^*A-\right.$$
$$\left.-ZD^*DZ_1^*-ZD^*C-C^*DZ_1^*-C^*C\right](BZ_1^*+A)^{-1}=$$
$$=\left(ZB^*+A^*\right)^{-1}\left[Z\left(B^*B-D^*D\right)Z_1^*+Z\left(B^*A-D^*C\right)+\right.$$
$$\left.+\left(A^*B-C^*D\right)Z_1^*+A^*A-C^*C\right]\left(BZ_1^*+A\right)^{-1}=$$
$$=(ZB^*+A^*)^{-1}(I-ZZ_1^*)(BZ_1^*+A)^{-1}. \qquad (6)$$

tengliklarga ega bo‘lamiz.

Agar,

$$\Delta_z=(I-ZZ^*)\bar{\partial}_Z(I-Z^*Z)\partial_Z{}',$$

deb belgilash kiritsak, biz quyidagiga ega bo‘lamiz:

$$\Delta_w=(I-WW^*)\bar{\partial}_W(I-W^*W)\partial'_W=$$
$$=(ZB^*+A^*)^{-1}(I-ZZ^*)(BZ^*+A)^{-1}(BZ^*+A)\bar{\partial}_Z$$
$$=(Z^*C^*+D^*)(Z^*C^*+D^*)^{-1}(I-Z^*Z)$$
$$(CZ+D)^{-1}(CZ+D)\partial'_Z(ZB^*+A^*)=$$
$$=(ZB^*+A^*)^{-1}\Delta_Z(ZB^*+A^*).$$

**Ta’rif 1** [3:117]. Operator Sp $\Delta_Z$ - $\mathfrak{R}_1$ klassik sohaning Laplas operatori deyiladi.

Bunda Sp - ifoda $\Delta_Z$ matritsaning diagonal elementlari yig‘indisi.

E’tibor berish kerakki, $\bar{\partial}_Z(I-Z^*Z)$ ifodasi faqat matritsalarni formal ko‘paytirishni anglatadi, ya’ni $(I-Z^*Z)$ matritsa elementlari differensiallanmaydi. Sp $\Delta_Z$ operatorining batafsil yozuvi quyidagi ko‘rinishga ega [(4)]:

$$Sp\Delta_Z=\sum_{i,j=1}^{m}\sum_{\beta,\gamma=1}^{n}\left(\delta_{ij}-\sum_{\alpha=1}^{n}z_{i\alpha}\bar{z}_{j\alpha}\right)$$
$$\left(\delta_{\beta\gamma}-\sum_{k=1}^{n}\bar{z}_{k\beta}z_{k\gamma}\right)\frac{\partial^2}{\partial z_{i\gamma}\partial\bar{z}_{j\beta}}.$$

Bu ifoda Laplas operatorining (2) ko‘rinishidagi o‘zgarishlar orqali invariant ekanligini, ya’ni $\mathfrak{R}_1$ sohasidagi akslantirishlar guruhiga nisbatan invariantligini ko’rsatadi.

**Ta’rif 2.** Agar haqiqiy qiymatli $u(Z)$ funksiyasi $Sp\Delta_Z\cdot u$ =0 tenglamani qanoatlandirsa, u holda $u(Z)$ funksiyasi $\mathfrak{R}_1$ sohasida garmonik funksiya deb ataladi.

Ravshanki, garmoniklik xossasi $\Gamma_1$ sistemaga nisbatan invariant. Garmonik funksiyalar to‘plami chiziqli.

**Ta’rif 3.** $\bar{\mathfrak{R}}_1$ da $G_I$ da uzluksiz bo‘lgan garmonik funksiyalar to‘plamini $\mathfrak{H}$ sinf deb ataladi.

Endi klassik saholar uchun Puasson yadrosini ko‘rib chiqamiz [3:98].

$U\in G$ bo’lsa, $\mathfrak{R}_1$ uchun

$$\mathrm{P}\left(Z,U\right)=\frac{1}{V\left(G_I\right)}\cdot\frac{\left[\det(I-ZZ^*)\right]^n}{\left|\det(I-ZU^*)\right|^{2n}}.$$

Bunda $V\left(G_I\right)$- [3:93] bo‘yicha birinchi tip klassik sohaning ostovi hajmi va u quyidagi korinishda ifodalanadi:

$$V\left(G_I\right)=\frac{\left(2\pi\right)^{\frac{n(n+1)}{2}}}{1!2!...\left(n-1\right)!}.$$

Xususan, m=n bo‘sa quyidagicha yozish mumkin

$$\mathrm{P}\left(Z,U\right)=\frac{1}{V\left(G_I\right)}\cdot\frac{\left[\det(I-ZZ^*)\right]^n}{\left|\det(Z-U)\right|^{2n}}.$$

Agar m=1 bo‘lsa $C^n$ kompleks fazodagi Puasson yadrosi kelib chiqadi [5].

**Teorema 1.** Agar $G_I$ dagi $U$ ni (2) akslantirish yordamida $V$ ga ($V\in G_I$) akslantirilsa, u holda

$$P_1(W,V)=P_1(Z,U)\left|\det(BU^*+A)\right|^{2n}.$$

tenglik orinli.

**Isboti.** (5) va (6) dan kelib chiqib,

$$W=\left(ZB^*+A^*\right)^{-1}\left(ZD^*+C^*\right),\ V=\left(UB^*+A^*\right)^{-1}\left(UD^*+C^*\right),$$

$$I-WV^*=I-\left(ZB^*+A^*\right)^{-1}\left(ZD^*+C^*\right)$$
$$\left(DU^*+C\right)\left(BU^*+A\right)^{-1}=$$
$$=\left(ZB^*+A^*\right)^{-1}\left[\left(ZB^*+A^*\right)\left(BU^*+A\right)-\right.$$
$$\left.-\left(ZD^*+C^*\right)\left(DU^*+C\right)\right]\left(BU^*+A\right)^{-1}=$$
$$=\left(ZB^*+A^*\right)^{-1}\left[ZB^*BU^*+ZB^*A+A^*BU^*+\right.$$
$$\left.+A^*A-ZD^*DU^*-ZD^*C-C^*DU^*-C^*C\right]\left(BU^*+A\right)^{-1}=$$
$$=\left(ZB^*+A^*\right)^{-1}\left[Z\left(B^*B-D^*D\right)U^*+\right.$$
$$\left.+Z\left(B^*A-D^*C\right)+\left(A^*B-C^*D\right)U+A^*A-C^*C\right]$$
$$\left(BU^*+A\right)^{-1}=$$
$$=\left(ZB^*+A^*\right)^{-1}\left(I-ZU^*\right)\left(BU^*+A\right)^{-1}.$$
$$\left(I-WV^*\right)^{-1}=\left(BU^*+A\right)\left(I-ZU^*\right)^{-1}\left(ZB^*+A^*\right).$$
$$I-VW^*=I-\left(UB^*+A^*\right)^{-1}\left(UD^*+C^*\right)$$
$$\left(DZ^*+C\right)\left(BZ^*+A\right)^{-1}=$$
$$=\left(UB^*+A^*\right)^{-1}\left[\left(UB^*+A^*\right)\left(BZ^*+A\right)-\right.$$
$$\left.-\left(UD^*+C^*\right)\left(DZ^*+C\right)\right]\left(BZ^*+A\right)^{-1}=$$
$$=\left(UB^*+A^*\right)^{-1}\left[UB^*BZ^*+A^*BZ^*+UB^*A+\right.$$
$$\left.+A^*A-UD^*DZ^*-UD^*C-C^*DZ^*-C^*C\right]\left(BZ^*+A\right)^{-1}=$$
$$=\left(UB^*+A^*\right)^{-1}\left[U\left(B^*B-D^*D\right)Z^*+\left(A^*B-C^*D\right)Z^*+\right.$$
$$\left.+U^*\left(B^*A-D^*C\right)+A^*A-C^*C\right]\left(BZ^*+A\right)^{-1}=$$
$$=\left(UB^*+A^*\right)^{-1}\left(I-UZ^*\right)\left(BZ^*+A\right)^{-1}.$$
$$\left(I-VW^*\right)^{-1}=\left(BZ^*+A\right)\left(I-UZ^*\right)^{-1}\left(UB^*+A^*\right).$$
$$\left(I-WV^*\right)^{-1}\left(I-WW^*\right)\left(I-VW^*\right)^{-1}=$$
$$=\left(BU^*+A\right)\left(I-ZU^*\right)^{-1}\left(ZB^*+A^*\right)\left(ZB^*+A^*\right)^{-1}$$
$$\left(I-ZZ^*\right)\left(BZ^*+A\right)^{-1}\left(BZ^*+A\right)\left(I-UZ^*\right)^{-1}\left(UB^*+A^*\right)=$$
$$=\left(BU^*+A\right)\left(I-ZU^*\right)^{-1}\left(I-UZ^*\right)^{-1}\left(UB^*+A^*\right).$$

Demak,

$$\frac{\left[\det\left(I-WW^*\right)\right]^n}{\left|\det\left(I-WV^*\right)\right|^{2n}}=\frac{\left[\det\left(I-ZZ^*\right)\right]^n}{\left|\det\left(I-ZU^*\right)\right|^{2n}}\left|\det\left(BU^*+A\right)\right|^{2n}.$$

Agar m=1 bo‘sa $C^n$ kompleks fazodagi [2:36, (2) tenglik] munosabatga ega bo‘lamiz.

**Teorema 2.** $P_1(Z,U)$- Puasson yadrosi $\mathfrak{R}_1$ sohada $G$ sinfga tegishli emas garmonik funksiya boladi.

**Isboti**. Dastlab $Z$=0 bo‘lganda $\left(Sp\Delta_Z\right)P_1(Z,U)=0$ bo’lishini isbotlaymiz.

$$Z=\left(z_{j\alpha}\right),\qquad U=\left(u_{j\alpha}\right),\qquad 1\le j\le m,\quad 1\le\alpha\le n.$$

$$\left(Sp\Delta_Z\right)P_1\left(Z,U\right)\Big|_{Z=0}=Sp\left(\overline{\partial}_Z\cdot\partial'_Z\right)\cdot P_1\left(Z,U\right)\Big|_{Z=0}=$$
$$=\left[\sum_{j=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial\overline{z}_{j\alpha}\partial z_{j\alpha}}P_1\left(Z,U\right)\right]_{Z=0}=$$
$$=\frac{1}{V\left(\mathfrak{C}_1\right)}\bullet\sum_{j=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial\overline{z}_{j\alpha}\partial z_{j\alpha}}\frac{\left[\det\left(I-ZZ^*\right)\right]^n}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^n}\Bigg|_{Z=0}=$$
$$=\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\sum_{j=1}^{m}\sum_{\alpha=1}^{n}\left\{\frac{\partial^2}{\partial\overline{z}_{j\alpha}\partial z_{j\alpha}}\left[\det\left(I-ZZ^*\right)\right]^n+\right.$$
$$\left.+\frac{\partial}{\partial z_{j\alpha}}\det\left(I-ZU^*\right)^{-n}\frac{\partial}{\partial\overline{z}_{j\alpha}}\left[\det\left(I-Z^*U\right)\right]^{-n}\right\}=$$
$$=\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\sum_{j=1}^{m}\sum_{\alpha=1}^{n}\left\{\frac{\partial^2}{\partial\overline{z}_{j\alpha}\partial z_{j\alpha}}\left(1-n\sum_{\beta=1}^{n}\sum_{k=1}^{m}\left|z_{k\beta}\right|^2\right)+\right.$$
$$\left.+\frac{\partial}{\partial z_{j\alpha}}\left(1+n\sum_{\beta=1}^{n}\sum_{k=1}^{m}z_{k\beta}\overline{u}_{k\beta}\right)\frac{\partial}{\partial\overline{z}_{j\alpha}}\left(1+n\sum_{\beta=1}^{n}\sum_{k=1}^{m}\overline{z}_{k\beta}u_{k\beta}\right)\right\}\Bigg|_{Z=0}=0.$$

Endi bu teoremani xususiy m=2, n=2 holatini ko‘rib chiqamiz.

Bizga $Z=\begin{pmatrix}z_{11}&z_{12}\\z_{21}&z_{22}\end{pmatrix}$ va $UU^*=I$ shartni qanoatlantiruvchi $U=\begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix}$ matritsalari berilgan bo‘lsin.

$$\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\frac{\partial^2}{\partial\overline{z}_{j\alpha}\partial z_{j\alpha}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\Bigg|_{Z=0}=$$
$$=\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\left\{\frac{\partial^2}{\partial\overline{z}_{11}\partial z_{11}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\right|_{Z=0}+$$
$$+\frac{\partial^2}{\partial\overline{z}_{12}\partial z_{12}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\Bigg|_{Z=0}+$$
$$+\frac{\partial^2}{\partial\overline{z}_{21}\partial z_{21}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\Bigg|_{Z=0}+$$
$$\left.+\frac{\partial^2}{\partial\overline{z}_{22}\partial z_{22}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\Bigg|_{Z=0}\right\}.$$

Hisob-kitoblarni soddalashtirish maqsadida biz Z=0 nuqtadagi garmoniklik shartini tekshiramiz. Ma’lumki, Puasson yadrosining Z=0 nuqtadagi Laplas operatorini hisoblaganda, funksiyaning Teylor yoyilmasidagi to‘rtinchi va undan yuqori tartibli hadlari, shuningdek, faqat $Z$ yoki $\overline{Z}$ ga nisbatan chiziqli bo‘lgan hadlar ikkinchi tartibli differensiallash va Z=0 qiymatni qo‘yish natijasida nolga aylanadi.

$$\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left(\left[\det\left(I-ZZ^*\right)\right]^2\right)\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[\det\begin{pmatrix}1-\left(|z_{11}|^2+|z_{12}|^2\right) & -\left(z_{11}\overline{z}_{21}+z_{12}\overline{z}_{22}\right)\\ -\left(z_{21}\overline{z}_{11}+z_{22}\overline{z}_{12}\right) & 1-\left(|z_{21}|^2+|z_{22}|^2\right)\end{pmatrix}\right]^2\Bigg|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[\left[1-\left(|z_{11}|^2+|z_{12}|^2\right)\right]\right.$$

$$\left.\left[1-\left(|z_{21}|^2+|z_{22}|^2\right)\right]-\left(z_{11}\overline{z}_{21}+z_{12}\overline{z}_{22}\right)^2\right]^2{}_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[1-\left(|z_{11}|^2+|z_{12}|^2+|z_{21}|^2+|z_{22}|^2\right)\right]^2\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[1-2\left(|z_{11}|^2+|z_{12}|^2+|z_{21}|^2+|z_{22}|^2\right)\right]\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left(1-2\sum_{\beta,k=1}^{2}z_{k\beta}\overline{z}_{k\beta}\right)\Bigg|_{Z=0}$$

$$\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[\det\left(I-ZU^*\right)\right]^{-2}\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[1+2\left(z_{11}\overline{u}_{11}+z_{12}\overline{u}_{12}+z_{21}\overline{u}_{21}+z_{22}\overline{u}_{22}\right)\right]\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[1+2\sum_{\beta,k=1}^{2}z_{k\beta}\overline{u}_{k\beta}\right]\Bigg|_{Z=0}$$

$$\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[\det\left(I-UZ^*\right)\right]^{-2}\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left[1+2\left(u_{11}\overline{z}_{11}+u_{12}\overline{z}_{12}+u_{21}\overline{z}_{21}+u_{22}\overline{z}_{22}\right)\right]\Big|_{Z=0}=$$

$$=\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left(1+2\sum_{k,\beta=1}^{2}u_{k\beta}\overline{z}_{k\beta}\right)\Bigg|_{Z=0}$$

Natijada quyidagi tenglikka ega bo‘lamiz:

$$\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\frac{\left[\det\left(I-ZZ^*\right)\right]^2}{\left[\det\left(I-ZU^*\right)\left(I-UZ^*\right)\right]^2}\Bigg|_{Z=0}=$$

$$=\frac{1}{V\left(\mathfrak{C}_1\right)}\cdot\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\left\{\frac{\partial^2}{\partial \overline{z}_{j\alpha}\partial z_{j\alpha}}\left(1-2\sum_{\beta,k=1}^{2}\left|z_{k\beta}\right|^2\right)+\right.$$

$$\left.+\frac{\partial}{\partial z_{j\alpha}}\left(1+2\sum_{\beta,k=1}^{2}z_{k\beta}\overline{u}_{k\beta}\right)\frac{\partial}{\partial \overline{z}_{j\alpha}}\left(1+2\sum_{k,\beta=1}^{2}u_{k\beta}\overline{z}_{k\beta}\right)\right\}\Bigg|_{Z=0}=$$

$$=-8+4\left(\overline{u}_{11}+\overline{u}_{12}+\overline{u}_{21}+\overline{u}_{22}\right)\left(u_{11}+u_{12}+u_{21}+u_{22}\right)=0.$$

$\Re_1$ sohaning $\Gamma_1$ sistemaga nisbatan tranzitivligi va 1-teoremaga asosan $\Re_1$ sohaning ixtiyoriy ichki nuqtasi uchun teoremaning tasdig‘in olamiz. 2-teorema isbotlandi.

**Xulosa.** Ushbu maqolada birinchi turdagi $\Re_1$ klassik sohalari uchun differensial operatorlar, xususan, Beltrami-Laplas operatorining matritsaviy ko‘rinishi va uning izi $\left(Sp\Delta_Z\right)$ atroflicha tadqiq qilindi. Tadqiqot davomida Puasson yadrosining garmoniklik xossasini isbotlash uchun yadro funksiyasi murakkab matritsaviy differensiallash qoidalari asosida tahlil qilindi. n=2 xususiy holatida amalga oshirilgan hadma-had differensiallash jarayoni shuni ko‘rsatdiki, operator tarkibidagi metrika elementlari va yadro yoyilmasidagi kvadratik hadlar o‘zaro muvofiq kelib, Puasson yadrosining garmoniklik shartini to‘liq qanoatlantiradi. Olingan natijalar Puasson yadro teoremasini nafaqat nazariy jihatdan tasdiqlaydi, balki matritsaviy sohalarda Dirixle masalasini yechishda differensial operatorlarning fundamental mexanizmini ochib beradi.

## Adabiyotlar


1. Мурнаган Ф.Д. Теория представлений групп. – Москва: «ИЛ», 1950.
2. Рудин У. Теория функций в единичном шаре из $\mathbb{C}^n$. – Москва: «Мир», 1984.
3. Хуа Ло-кен. Гармонический анализ функций многих комплексных переменных в классических областях. – Москва: «ИЛ», 1959.
4. Худайберганов Г., Кытманов А.М., Шаимкулов Б.А. Анализ в матричных областях. – Красноярск: СФУ, 2017.
5. Шабат Б.В. Введение в комплексный анализ. Ч.2. – Москва: «Наука», 1985.