THE SOLUTION TO A SYSTEM OF EQUATIONS OF THE HOPFTYPE IN THE FORM OF A TRAVELING WAVE

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РЕШЕНИЕ СИСТЕМЫ ПОЛУТИПНЫХ УРАВНЕНИЙ В ВИДЕ БЛУЖДАЮЩЕЙ ВОЛНЫ

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Tayanch so‘zlar: konvektiv had, nochiziqlik, Byurgers va Hopfa turidagi tenglamalar tizimlari, nochiziqli oddiy differensial tenglamalar.

Rezyume. Ikki tezlikli gidrodinamikaning nostatsionar tenglamalar tizimidan Byurgers va Hopf turidagi tenglamalar tizimi hosil qilindi. Energiya dissipasiyasi subsistemalar viskozligi va komponentlararo ishqalanish koeffitsienti hisobiga yuzaga kelishi e'tiborga olinishi kerak. Hopf turidagi model uchun differensial tenglamalar tizimining sayohat qiluvchi to‘lqinlar ko‘rinishidagi tahlili tizimni nochiziqli oddiy differensial tenglamalar sistemasini yechishga keltiradi.

Ключевые слова: конвективный член, нелинейность, системы уравнений типа Бюргерса и Хопфа, нелинейные обыкновенные дифференциальные уравнения.

Резюме. Из системы нестационарных уравнений двухскоростной гидродинамики нами получены системы уравнений типа Бюргерса и Хопфа. Следует учитывать, что рассеяние энергии происходит за счёт вязкостей подсистем и коэффициента межкомпонентного трения. Для модели типа Хопфа анализ системы дифференциальных уравнений в виде бегущих волн сводится к решению системы нелинейных обыкновенных дифференциальных уравнений.

Key words: convective term, nonlinearity, systems of equations of Burgers and Hopf type, nonlinear ordinary differential equations.

Summary. From the system of non-stationary equations of two-velocity hydrodynamics, a system of equations of the Burgers and Hopf type was obtained. It should be taken into account that energy dissipation occurs due to the viscosity of the subsystems and the coefficient of friction between the components. Analysis of the system of differential equations in the form of traveling waves for the Hopf-type model reduces the system to solving a system of nonlinear ordinary differential equations.

Introduction. Nonlinear wave processes frequently appear in physics and engineering, including liquid and gas turbulence, plasma motion, ocean waves, and acoustic fluctuations. Such systems exhibit complex behavior, making their analytical study difficult. Simplified mathematical models such as the Burgers and Hopf-type equations are used to describe essential nonlinear dynamics.

Weak turbulence occurs in weakly interacting dispersive systems, whereas strong turbulence forms through coherent wave interactions. To better understand nonlinear wave mechanisms, solvable model systems are required.

The Nature gives us many examples of nonlinear random fields and waves. A demand for studying them does not need any comments and has become clear from a brief listing. These are liquid and gas turbulence, chaotic plasma motions, intense acoustic noise, random sea surface waves, etc. One of the trends in the development of the theory of nonlinear random waves is the selection of a small number of concepts and ideas that allow, on the one hand, to unify the behavior of nonlinear random waves of different physical nature, and on the other hand, I am pleased to clearly classify nonlinear accidental waves according to the nature of their interactions. These primarily include the concepts of weak and strong turbulence. The weak turbulence regime is characteristic of weakly linear waves in media with strong dispersion, when the interaction energy of spatial harmonics is small as compared to their total energy. The chaotic phase approximation, assuming that the interaction among harmonics is incoherent, allows one in this case to turn to a closed static description of the turbulent regime [1, 2]. If the dispersion of the waves is small or absent, the properties turbulence are determined by strong interaction of a large number of coherent wave harmonics. In such cases, it is customary to speak about strong turbulence. A well-known most example of strong turbulence is the vortex turbulence of a liquid with low viscosity [3].

An extreme difficulty of analyzing nonlinear waves, especially, waves of strong turbulence, has led to another development trend in their theory - the transition from complex equations of nonlinear accidental waves to simpler model equations [4]. One of such model equations of turbulence is the Burgers equation. Modern theories of continuum mechanics [5] - [7] suggest that a medium possesses its prehistory and in the general case a material can have an arbitrarily long "memory". However, a long memory generates significant difficulties, which can be overcome in the two ways: first, by considering special classes of motions in which "memory" whatever it may be cannot significantly manifest itself (for example, viscometric flows of viscous fluids [5, Ch. V]), second, by distinguishing classes of media or materials in which stresses at any point are affected only by the background of motion in an arbitrarily small time interval. Materials of this type are called materials with in_nitesimal memory. The most important materials with infinitesimal memory are those in which stresses at the point x at the time instant $t*$ are determined by the first n derivatives of the deformation gradient $F(x)$ with respect to time t at the same time $t*$. Such materials are called materials of the differential type of complexity $n = 1, 2, \dots$. The theory of isotropic liquids of the differential type of complexity n was constructed by Rivlin and Eriken (see [5], [6]), and based on this theory a simpler asymptotic theory of slow motions of liquids of order $n = 0, 1, 2, \dots$. was constructed by Coleman and Noll (see [5]). In the case of incompressible fluids, a zero-order fluid is an elastic fluid whose motion is described by the Euler equations.

$$\frac{\partial \mathrm{v}}{\partial t} + v_k \frac{\partial \mathrm{v}}{\partial x_k} + \frac{1}{\rho}\nabla p = \mathbf{F}, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

first order fluid is a Newtonian linearly viscous _uid whose motion is described by the Navier-Stokes equations

$$\frac{\partial \mathrm{v}}{\partial t} + v_k \frac{\partial \mathrm{v}}{\partial x_k} - \nu \nabla \mathrm{v} + \frac{1}{\rho}\nabla p = \mathbf{F}, \quad \operatorname{div} \mathbf{v} = 0, \tag{2}$$

moreover, $\rho > 0$ and $v > 0$.

A system of equations of the two- velocity hydrodynamics. The equations of motion of a two-velocity medium in the dissipative case with one pressure in the system in the isothermal case have the form [8-10]

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathrm{v}_1) = 0\,, \tag{3}$$

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathrm{v}_2) = 0, \tag{4}$$

$$\frac{\partial \mathrm{v}_1}{\partial t} + (\mathrm{v}_1, \nabla)\mathrm{v}_1 = -\frac{1}{\rho}\nabla p + \frac{\nu_1}{\rho}\Delta \mathrm{v}_1 \frac{\nu_1 + 3\mu_1}{3\,\rho}\nabla \operatorname{div} \mathbf{v_1} + \frac{\rho_2}{2\,\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, \tag{5}$$

$$\frac{\partial \mathrm{v}_2}{\partial t} + (\mathrm{v}_2, \nabla)\mathrm{v}_2 = -\frac{1}{\rho}\nabla p + \frac{\nu_2}{\rho}\Delta \mathrm{v}_2 \frac{\nu_2 + 3\mu_2}{3\,\rho}\nabla \operatorname{div} \mathbf{v_2} + \frac{\rho_1}{2\,\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, \tag{6}$$

where v1 and v2 are the velocity vectors of the subsystems that make up a two- velocity continuum with the corresponding partial densities ρ_1 and ρ_2, $\nu_1(\mu_1)$ and $\nu_2(\mu_2)$ are the respective shear (bulk) viscosities, $\rho = \rho_1 + \rho_2$ is the total density of the two- velocity continuum; **F** is the mass vector force referred to as a unit of mass. The system of equations (3) - (6) is closed by the equation of state of the two- velocity continuum $p = p(\rho, (\mathrm{v}_1 - \mathrm{v}_2)^2)$.

We rewrite equations (5) and (6) in the equivalent form

$$\rho\left(\frac{\partial \mathrm{v}_1}{\partial t} + \frac{1}{2}\nabla(v_1^2) - \mathrm{v}_1 \times \operatorname{rot} \mathbf{v}_1\right) = = -\nabla p + \nu_1\,\Delta \mathrm{v}_1 + \left(\frac{\nu_1}{3} + \mu_1\right)\nabla \operatorname{div} \mathrm{v}_1 + \frac{\rho_2}{2}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \rho\mathbf{F}, \tag{7}$$

$$\rho\left(\frac{\partial \mathrm{v}_2}{\partial t} + \frac{1}{2}\nabla(v_2^2) - \mathrm{v}_2 \times \operatorname{rot} \mathbf{v}_2\right) = -\nabla p + \nu_2\,\Delta \mathrm{v}_2 + \left(\frac{\nu_2}{3} + \mu_2\right)\nabla \operatorname{div} \mathrm{v}_2 + \frac{\rho_1}{2}\nabla(\mathrm{v}_2 - \mathrm{v}_2)^2 + \rho\mathbf{F}, \tag{8}$$

From these equations, we can derive other ones that would determine a change in vortices in the course of time. To this end, we apply the operator rot to both sides of equations (7), (8). As a result, we obtain

$$\frac{\partial \Omega_1}{\partial t} - \operatorname{rot}(\mathrm{v}_1 \times \Omega_1) = -\operatorname{rot}\left(\frac{\nabla \mathrm{p}}{\rho}\right) + \mathrm{v}_1 \Delta \Omega_1 + + \operatorname{rot}\left(\frac{\frac{\nu_1}{3} + \mu_1}{\rho}\nabla \operatorname{div} \mathrm{v}_1\right) + \operatorname{rot}\left(\frac{\rho_2}{2\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2\right) + \operatorname{rot}\mathbf{F},$$

$$\frac{\partial \Omega_2}{\partial t} - \operatorname{rot}(\mathrm{v}_2 \times \Omega_2) = -\operatorname{rot}\left(\frac{\nabla \mathrm{p}}{\rho}\right) + \mathrm{v}_2 \Delta \Omega_2 + + \operatorname{rot}\left(\frac{\frac{\nu_2}{3} + \mu_2}{\rho}\nabla \operatorname{div} \mathrm{v}_2\right) + \operatorname{rot}\left(\frac{\rho_1}{2\rho}\nabla(\mathrm{v}_2 - \mathrm{v}_2)^2\right) + \operatorname{rot}\mathbf{F},$$

A system of equations for the two-velocity hydrodynamics of a compressible medium in the case of constant saturation in the reversible hydrodynamic case has the form

$$\operatorname{div} \mathrm{v}_1 = 0\,,\ \operatorname{div} \mathrm{v}_2 = 0, \qquad (9)$$

$$\frac{\partial \mathrm{v}_1}{\partial t} + v_{1k}\frac{\partial \mathrm{v}_1}{\partial x_k} = -\frac{1}{\rho}\nabla p + \frac{\rho_2}{\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, \quad (10)$$

$$\frac{\partial \mathrm{v}_2}{\partial t} + v_{2k}\frac{\partial \mathrm{v}_2}{\partial x_k} = -\frac{1}{\rho}\nabla p + \frac{\rho_1}{\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, \quad (11)$$

while in the dissipative case, it has the form

$$\operatorname{div} \mathrm{v}_1 = 0\,,\ \operatorname{div} \mathrm{v}_2 = 0, \qquad (12)$$

$$\frac{\partial \mathrm{v}_1}{\partial t} + v_{1k}\frac{\partial \mathrm{v}_1}{\partial x_k} = -\frac{1}{\rho}\nabla p + \eta_1\Delta\mathrm{v}_1 + \frac{\rho_2}{\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, (13)$$

$$\frac{\partial \mathrm{v}_2}{\partial t} + v_{2k}\frac{\partial \mathrm{v}_2}{\partial x_k} = -\frac{1}{\rho}\nabla p + \eta_2\Delta\mathrm{v}_2 + \frac{\rho_1}{\rho}\nabla(\mathrm{v}_1 - \mathrm{v}_2)^2 + \mathbf{F}, (14)$$

In formulas (13) and (14), the kinematic viscosities of subsystems are denoted by
$\eta_k, (k = 1, 2)$.

The Burgers-type system of equations. The systems of equations (9) - (11) and (12) - (14) are a generalization of systems (1) and (2) for a multiphase medium, respectively. A subclass of system (12) - (14) are the Burgers-type systems of equations. In the one-dimensional case, this system has the form

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} = \eta_1\frac{\partial^2 v_1}{\partial x^2} - \tilde{b}\,(v_1 - v_2) + \mathbf{F}, \quad (15)$$

$$\frac{\partial v_2}{\partial t} + v_2\frac{\partial v_2}{\partial x} = \eta_2\frac{\partial^2 v_2}{\partial x^2} - b\,(v_1 - v_2) + \mathbf{F}, \quad 16)$$

In formulas (15) and (16) $\tilde{b} = b\frac{\rho_2}{\rho_1}$ is the interfacial friction coefficient, which is an analog of the Darcy coefficient for porous media.

The Hopf-type system of equations. System (15), (16) differs from the system of the two-velocity hydrodynamics in the dissipative case due to the presence of the friction coefficient, the absence of pressure and the incompressibility condition. For this reason, problems associated with the Burgers type system are sometimes referred to as two- velocity, non pressure hydrodynamics. The case $\eta_1 = 0\,,\ \eta_2 = 0$ will be called an inviscid system of the Burgers type or a system of the Hopf type, or a system of the Riemann type, which gives a simple quasilinear system of equations

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} + \tilde{b}\,(v_1 - v_2) + \mathbf{F}, \qquad (17)$$

$$\frac{\partial v_2}{\partial t} + v_2\frac{\partial v_2}{\partial x} + b\,(v_1 - v_2) + \mathbf{F}, \qquad (18)$$

Zeldovich has proposed considering an inviscid free system in the one velocity case in the absence of mass forces ($F = 0$) as an equation describing the evolution of a rare_ed gas of non-interacting particles [10]. According to the idea, proposed the pure kinematics of the underlying particles can lead to features in the distribution of mass and become responsible for the heterogeneity of matter in the universe [8].

The solution to a system of equations of the Hopftype in the form of a traveling wave.

We seek a solution to system (17), (18) ($F = 0$) in the form of a traveling wave. We introduce a new variable $\xi = x = ct$ and

$$v_1(x,t) = U_1(\xi),\ v_2(x,t) = U_2(\xi), \qquad (19)$$

where c is the velocity of a unidirectional traveling wave along the axis Ox at the time instant t. Thus, the functions $U1(\ \xi), U2(\ \xi)$ satisfy a system of ordinary differential equations of the form

$$-cU_1' + U_1U_1' = -b(U_1 - U_2),\ -cU_2' + U_2U_2' = \varepsilon b(U_1 - U_2).$$

Hence, after carrying out simple transformations with respect to the $\widetilde{U_1} = (U_1 - c)^2,\ \widetilde{U_2} = (U_2 - c)^2$ we come to a system of ordinary differential equations:

$$\widetilde{U_1'} = -2b\left(\sqrt{\widetilde{U_1}} - \sqrt{\widetilde{U_2}}\right), \qquad (20)$$

$$\widetilde{U_2'} = 2\varepsilon b\left(\sqrt{\widetilde{U}_1} - \sqrt{\widetilde{U_2}}\right), \qquad (21)$$

Multiplying both sides of equation (20) by " and adding the resulting equation to equation (21), we obtain

$$(\varepsilon\widetilde{U}_1 + \widetilde{U}_2)' = 0.$$

From here, after conducting integration, we obtain the relationship between the functions $\widetilde{U}_1$ and $\widetilde{U}_2$:

$$\varepsilon\widetilde{U}_1 + \widetilde{U}_2 = C_1 \qquad (22)$$

where $C1$ is the positive integration constant.[9].

From (20), after exclusion of $\widetilde{U}_2$ and taking into account (22), with respect to $\widetilde{U}_1$ we obtain the ordinary differential equation of the first order

$$\widetilde{U}'_1 = 2b\left(\sqrt{C_1 - \varepsilon\widetilde{U}_1} - \sqrt{\widetilde{U}_1}\right).$$

The solution, which has the form

$$\frac{1}{(1+\varepsilon)^{\frac{3}{2}}}\left(-2\sqrt{1+\varepsilon}\left(\sqrt{\widetilde{U}_1} + \sqrt{C_1 - \varepsilon\widetilde{U}_1}\right) + \sqrt{C_1}\left(2\mathrm{ArcTanh}\left[\frac{\sqrt{\widetilde{U}_1\ \sqrt{1+\varepsilon}}}{\sqrt{C_1}}\right] - \ln\left[C_1\left(\left(\sqrt{C_1\sqrt{1+\varepsilon} + \sqrt{C_1 - \widetilde{U}_1\varepsilon}}\right)^2 - \widetilde{U}_1\varepsilon_2\right)\right]\right)\right) = 2b\xi + C_2, \qquad (23)$$

where $C2$ is the integration constant.[10].

After some transformations relations (23) takes the form:

$$\sqrt{\widetilde{U}_1} + \sqrt{C_1 - \varepsilon\widetilde{U}_1} - \frac{\sqrt{C_1}}{2\sqrt{1+\varepsilon}}\left(2\mathrm{ArcTanh}\left[\frac{\sqrt{\widetilde{U}_1\ \sqrt{1+\varepsilon}}}{\sqrt{C_1}}\right] - \ln\left[C_1\left(\left(\sqrt{C_1\sqrt{1+\varepsilon} + \sqrt{C_1 - \widetilde{U}_1\varepsilon}}\right)^2 - \widetilde{U}\varepsilon_2\right)\right]\right) = -b(1+\varepsilon)\xi + C_2.$$

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Conclusion. A traveling-wave analytical solution for a Hopf-type nonlinear hydrodynamic system was obtained. The derived equation characterizes nonlinear wave propagation, showing how kinetic parameters determine wave speed, smoothness, and structure.

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