REZYUME. Maqolada plazmali nitridlash (azotlash bilan po'latning yuza qatlam o'zgarishi) borasida olib borilgan. O'tkazilgan metallografik sinovlar natijasida, yupqa, g'ovak va geterogen azotli qatlam mavjudligi yoki uning yo'qligi, yadrosi uning to'yinmagan holati va termal parametrlari o'zgarganligi aniqlangan. Nitridlash parametrlarini o'zgartirishda gazlar, vaqt va nitridlashning harorati o'zgartirish bo'yicha jarayon taklif qilindi. Ion azotlash jarayonining eng optimal natija olish uchun gaz oqimi, vaqt, harorat parametrlarini rostlash zarur.

РЕЗЮМЕ. В статье рассматривалось плазменное нитрование (изменение поверхностного слоя стали азотированием). Проведенные металлографические испытания выявили наличие или отсутствие тонкого, пористого и неоднородного азотистого слоя, ядра в его ненасыщенном состоянии и изменение тепловых параметров. При изменении параметров азотирования был предложен процесс изменения газов, времени и температуры азотирования. Для получения оптимального результата процесса ионного азотирования необходимо отрегулировать параметры расхода газа, времени, температуры.

SUMMARY. Plasma nitration (changing the surface layer of steel by nitriding) was considered in the article. The metallographic tests carried out revealed the presence or absence of a thin, porous and heterogeneous nitrogen layer, a core in its unsaturated state and a change in thermal parameters. When changing the nitriding parameters, the process of changing gases, time and temperature of nitriding was proposed. To obtain the optimal result of the ion nitriding process, it is necessary to adjust the parameters of gas flow, time, and temperature.

INTERVAL EIGENVALUE PROBLEM AND METHODS FOR ESTIMATING SOLUTION SETS

D.N.Khamroeva – *doctor of philosophy in physical and mathematical sciences*
Navoi state university

Tayanch so'zlar: interval tahlil, xos qiymat, simmetrik, interval matritsa, ichki baholash, tashqi baholash.

Ключевые слова: интервальный анализ, собственное значение, симметричное, интервальная матрица, внутренняя оценка, внешняя оценка.

Key words: interval analysis, eigenvalue, symmetric, interval matrix, internal evaluation, external evaluation.

Introduction. In many practical problems, it is necessary to take into account the inaccuracies or uncertainties of the numerical information being processed. For this purpose, there are currently such uncertainty models as, for example, probabilistic (stochastic), interval (limited in magnitude) and fuzzy (blur), which have found wide application in solving various applied problems. Among them, the interval description of uncertainty is the most “sparse”, but the mathematical apparatus for its processing is quite developed. A review of the world literature indicates that interest in interval analysis methods has been increasing intensively recently. The basics of interval analysis are covered in monographs by G.G.Alefeld [1], Yu.I.Shokin [2], S.A.Kalmykov [2], Z.Kh.Yuldashev [2], S.P.Shary [3], R.Moore [4], R.B.Keafott, Z.Rump, A.Neumayer [5], G.Mayer, E.Hansen and others.

Preliminary facts and notations. The interval algebraic eigenvalue problem arises in those problems where the value of the elements of a given matrix fluctuates with a known amplitude, i.e. is limited from below and above by certain numbers, in the form of intervals. In this case, the analysis is applied to interval matrices to answer three main questions [6]:

1. What is the arrangement of the eigenvalues of an interval matrix?
2. How does the spectrum of an interval matrix depend on the spectrum of its point matrices?
3. How can we calculate the tight lower and upper bounds for each eigenpair of an interval matrix?

One of the typical problems of interval analysis, which is closely related to the solution of the eigenvalue problem, is the problem of solving interval systems of linear algebraic equations (ISLAE).

By interval linear system

$$\boldsymbol{A}x = \boldsymbol{b} \tag{1}$$

we mean the family of all systems of linear equations $Ax = b$, with the condition $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$.

Here it is assumed that the matrix elements and the components of the vector of the right parts lie in the given intervals. Such an assumption is possible in the case when the coefficients of the system lie within some permissible deviations or have some variations. Systems of the form (1) are often encountered in practice.

The set of solutions of ISLAU in most cases is formulated as

$$\Xi = \left\{ x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b) \right\}. \tag{2}$$

Set (2) is called the united set of solutions of ISLAU [3]. For the combined set of ISLAU solutions, we present the Oettli-Prager characterization [7].

Theorem 1. The vector $x \in \mathbb{R}^n$ belongs to the set of solutions Ξ of the system $\boldsymbol{A}x = \boldsymbol{b}$ if and only if

$$\left| \left(\operatorname{mid} \boldsymbol{A}\right) x - \operatorname{mid} \boldsymbol{b} \right| \le \operatorname{rad} \boldsymbol{A} \cdot |x| + \operatorname{rad} \boldsymbol{b}. \tag{3}$$

For ISLAU of the form (1), a large number of methods for obtaining lower and upper bounds for the vector have been developed [1], [2], [3], [4], [8]. Some developed algorithms, for example [4], [9], [10], also calculate the exact interval hull covering the sets Ξ .

Statement of the problem. *Algebraic eigenvalue problem with square interval matrix* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$:

$$\boldsymbol{A}x = \lambda x, \tag{4}$$

are a little-studied area of interval analysis, relative to ISLAU.

The set of all eigenvalues for system (4) in the general case constitutes a domain in the field of complex numbers $\mathbb{C}$, defined as the united set

$$\Lambda(\boldsymbol{A}) := \{\lambda \in \mathbb{C} \mid A \in \boldsymbol{A},\ x \ne 0,\ Ax = \lambda x\}, \tag{5}$$

and it is a compact set. If the interval matrix $\boldsymbol{A}$ has a special structure, stronger results can be obtained. This is especially true when the interval matrix $\boldsymbol{A}$ is symmetric, since in this case all its eigenvalues form continuous real intervals.

Thus, for the interval symmetric eigenvalue problem according, we can write the solution set as

$$\Lambda(\boldsymbol{A}^s) := \{\lambda \in \mathbb{R} \mid \exists A \in \boldsymbol{A}^s,\ \ \exists x \neq 0,\ \ Ax = \lambda x\}. \quad (6)$$

In interval analysis, two most popular problems are considered for estimating the set of solutions of interval systems of linear (also nonlinear) equations, i.e., problems of external and internal estimation [3]. In the general case, by interval estimation we will understand the replacement of the exact range of values by its interval estimate. Then, for the problem of estimating the set of eigenvalues, we will reformulate the definition of these concepts for an arbitrary interval matrix, since a symmetric matrix is a special case.

Definition 1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an arbitrary real interval matrix. Then, when solving an interval algebraic eigenvalue problem, we call:*

1. *external interval estimation,* when complex intervals $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_i^{\mathrm{Re}} + j\boldsymbol{\lambda}_i^{\mathrm{Im}} \in \mathbb{IC}$ are sought, their finite union encompassing from the outside the set of solutions Λ, such that $\bigcup_{i=1}^{n} \boldsymbol{\lambda}_i \supseteq \Lambda$.

2. *internal interval estimation,* when complex intervals $\boldsymbol{\mu}_i = \boldsymbol{\mu}_i^{\mathrm{Re}} + j \cdot \boldsymbol{\mu}_i^{\mathrm{Im}} \in \mathbb{IC}$ are sought, their finite union contained in the set of solutions Λ, such that $\bigcup_{i=1}^{n} \boldsymbol{\mu}_i \subseteq \Lambda$.

In other words, the external interval estimate of the set Λ is a set that contains the set Λ as a subset, and the internal interval estimate of the set Λ is any subset of the set Λ [11]. Of course, external and internal evaluation problems are often encountered in solving practically important interval problems, but they are not the only possible ones; there are other methods. For example, in most cases it is necessary to maximize or minimize a certain functional on a set of solutions, i.e. the problem can be solved as an optimization problem.

The calculation of the set of eigenvalues is directly reduced to the solution of the system

$$Ax = \lambda x,\quad \|x\| = 1,\quad A \in \boldsymbol{A},\quad \lambda \in \boldsymbol{\lambda}, \qquad (7)$$

where is any vector norm.

Solution method and convergence study. The interval algebraic eigenvalue problem with respect to the matrices under consideration can be divided into:

- problems for interval matrices with constraints (6), i.e. additional constraints are imposed on the elements (for example, symmetric, skew-symmetric, persymmetric, Toeplitz, Hankel, etc.);
- problems for an arbitrary interval matrix (5).

We study problem (7) only for symmetric and arbitrary (non-symmetric) real interval matrices. Partial and complete eigenvalue problems for interval matrices are also considered. In the partial problem, we will look for the boundaries of one interval or several intervals, and possibly calculate the corresponding interval eigenvectors. Most often, the problem of determining the boundary of the interval containing the largest (or smallest) eigenvalues of the matrices $A \in \boldsymbol{A}$ arises.

The first real attempt to solve the problem of interval eigenvalues was undertaken in 1987 by S.V.Hollot and A.S.Bartlett [12]. They proved that the eigenvalues of the interval matrix $\boldsymbol{A}$ can be limited by the roots of the so-called boundary polynomials. Such polynomials form the edges of the convex hull of the characteristic polynomials generated by the interval matrix $\boldsymbol{A}$. They also showed that the spectrum of an interval matrix with real eigenvalues is completely determined by the eigenvalues of its vertex matrices. S.H.Lin and his co-authors in [13], following well-known results on estimates of the eigenvalues of constant matrices, studied the relationship between the elements of an interval matrix and the location of its eigenvalues. T.Mori and H.Kokame [15] in 1994 proposed a new technique for estimating the location of eigenvalues along with the upper and lower bounds of each eigenvalue. This technique was developed to be applicable only to a certain class of interval matrices, characterized by the property that all eigenvalues are real for any linear combination of the interval matrices $A \in \boldsymbol{A}$:

$$\lambda_i(\underline{A}) - \rho(\bar{A} - \underline{A}) \le \lambda_i(A) \le \lambda_i(\bar{A}) + \rho(\bar{A} - \underline{A})$$

where ρ is the spectral radius.

On the other hand, based on the invariance properties of the components of the eigenvectors of the middle matrix of the interval matrix $\boldsymbol{A}$, A.Deif [16], restricting the sets (5), for a symmetric interval matrix, arrived at the estimate

$$\frac{x^T \cdot \mathrm{mid}\,\boldsymbol{A} \cdot x - |x|^T \cdot \mathrm{rad}\,\boldsymbol{A} \cdot |x|}{x^T x} \le \lambda \le \frac{x^T \cdot \mathrm{mid}\,\boldsymbol{A} \cdot x + |x|^T \cdot \mathrm{rad}\,\boldsymbol{A} \cdot |x|}{x^T x}. \quad (8)$$

For a tridiagonal symmetric interval matrix, J.C.Commercon in his work [17] proposed an efficient algorithm for determining the eigenvalues of a tridiagonal symmetric interval matrix of dimension $n \times n$. The proposed method is based on a modification of the Sturm algorithm used for point matrices by using certain relations of the Sturm sequence to obtain expressions that are free and can also be used in interval arithmetic calculations. A simple but very useful theorem, proved by J.Rohn [18], allows us to establish an outer bound for the entire set $\Lambda(\boldsymbol{A})$, performing a relatively small amount of computation.

Theorem 2. *[18; pp. 10-50]. Let* $\boldsymbol{A} = [\mathrm{mid}\,\boldsymbol{A} - \mathrm{rad}\,\boldsymbol{A},\ \mathrm{mid}\,\boldsymbol{A} + \mathrm{rad}\,\boldsymbol{A}]$ *be an interval square matrix. Then for each eigenvalue λ of each $A \in \boldsymbol{A}$ we have*

$$\lambda_{\min}(\mathrm{mid}\,\boldsymbol{A}') - \rho(\mathrm{rad}\,\boldsymbol{A}') \le \mathrm{Re}\,\lambda \le \lambda_{\max}(\mathrm{mid}\,\boldsymbol{A}') + \rho(\mathrm{rad}\,\boldsymbol{A}'), \quad (9)$$

$$\lambda_{\min}(\mathrm{mid}\,\boldsymbol{A}'') - \rho(\mathrm{rad}\,\boldsymbol{A}'') \le \mathrm{Im}\,\lambda \le \lambda_{\max}(\mathrm{mid}\,\boldsymbol{A}'') + \rho(\mathrm{rad}\,\boldsymbol{A}''), \quad (10)$$

where

$$\mathrm{mid}\,\boldsymbol{A}'_c = \frac{1}{2}\left(\mathrm{mid}\,\boldsymbol{A} + (\mathrm{mid}\,\boldsymbol{A})^T\right),$$

$$\mathrm{rad}\,\boldsymbol{A}' = \frac{1}{2}\left(\mathrm{rad}\,\boldsymbol{A} + (\mathrm{rad}\,\boldsymbol{A})^T\right),$$

$$\mathrm{mid}\,\boldsymbol{A}'' = \begin{pmatrix} 0 & \frac{1}{2}\left(\mathrm{mid}\,\boldsymbol{A} - (\mathrm{mid}\,\boldsymbol{A})^T\right) \\ \frac{1}{2}\left((\mathrm{mid}\,\boldsymbol{A})^T - \mathrm{mid}\,\boldsymbol{A}\right) & 0 \end{pmatrix},$$

$$\mathrm{rad}\,\boldsymbol{A}'' = \begin{pmatrix} 0 & \mathrm{rad}\,\boldsymbol{A}' \\ \mathrm{rad}\,\boldsymbol{A}' & 0 \end{pmatrix}.$$

This theorem can be applied to solve problem (7) for both symmetric and asymmetric interval matrices. There is another similar theorem of Rohn [8], approximating the outer boundaries of the set $\Lambda_i(\boldsymbol{A}^S)$ for a symmetric interval matrix and it can be considered as a consequence of the

Wielandt-Hoffman theorem. A complete proof is available in [18].

Theorem 3. *Let* $\boldsymbol{A}^S \in \mathbb{IR}^{n\times n}$, $n \in \mathbb{N}$, *be a symmetric interval matrix. Then for* $i = 1, 2, \ldots, n$ *and* $A \in \boldsymbol{A}^S$, *the following holds:*

$$\lambda_i(\operatorname{mid}\boldsymbol{A}) - \rho(\operatorname{rad}\boldsymbol{A}) \le \lambda_i(A) \le \lambda_i(\operatorname{mid}\boldsymbol{A}) + \rho(\operatorname{rad}\boldsymbol{A}).$$

Obviously, Theorem 3 is a special case of Theorem 2 if we consider only symmetric interval matrices. Practical use of the theorem involves computing all eigenvalues of a given interval matrix by computing the individual eigenvalues of the matrix $\operatorname{mid}\boldsymbol{A}$ and the spectral radius $\operatorname{rad}\boldsymbol{A}$. In some cases we are required to find the upper boundary point, i.e. to calculate the largest eigenvalue of a given interval matrix. In this case the following theorem may be useful [11].

Theorem 4. Let $\boldsymbol{A}^S \in \mathbb{IR}^{n\times n}$, $n \in \mathbb{N}$, be a symmetric interval matrix. Then $\bar{\lambda}_1(\boldsymbol{A}^s) \le \lambda_1(|\boldsymbol{A}|)$, where $\bar{\lambda}_1$ is the upper bound for the largest eigenvalue of $\boldsymbol{A}^S$.

One of the iterative methods for finding an exact outer approximation of the boundaries of the eigenvalues of an interval matrix is the filtering method developed by M.Hladik and his co-authors [20]. The method is based on calculating some initial approximation with subsequent iterative improvement at each step by “cutting off” parts of the intervals. This approach is suitable for both symmetric and asymmetric interval matrices. It is based on the following theorem, the proof of which is given in [20].

Conclusion. The interval algebraic eigenvalue problem is a relatively young and poorly studied area of interval analysis. Unfortunately, there is still no monograph fully devoted to these issues. This area needs to systematize knowledge based on the results obtained by the authors. But materials devoted to interval eigenvalue problems and their applications have been published in the form of scientific articles, dissertations, technical reports, etc.

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REZYUME. Ushbu maqolada interval xos qiymat muammosi va yechimlar to‘plamini baholash usullaridan faqat tashqi interval baholash masalalari qaralgan va bizni eng kichik kenglikdagi interval yechimlar qiziqtiradi.

РЕЗЮМЕ. В данной статье из интервальной задачи собственных значений и методов оценки множества решений рассматриваются только внешние интервальные задачи оценивания, а нас интересуют интервальные решения наименьшей ширины.

SUMMARY. In this article, only external interval estimation problems are considered from the interval eigenvalue problem and methods for estimating the set of solutions, and we are interested in interval solutions of the smallest width.