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REZYUME. Ushbu maqola harorat, namlik, karbonat angidrid, kislorod va etilen darajalari kabi muhim atrof-muhit parametrlarini kuzatish va tartibga solishda ilg‘or sensorlarning rolini o‘rganadi. IoT texnologiyasini integratsiyalashgan holda, ushbu sensorlar real vaqt rejimida ma’lumotlarni yig‘ish, masofadan turib monitoring qilish va avtomatlashtirilgan boshqaruv tizimlarini saqlangan mahsulot sifati va uzoq umr ko‘rishini ta’minlaydi.

РЕЗЮМЕ. В этой статье исследуется роль современных датчиков в мониторинге и регулировании основных параметров окружающей среды, таких как температура, влажность, уровни углекислого газа, кислорода и этилена. Благодаря интеграции технологии Интернета вещей эти датчики обеспечивают сбор данных в реальном времени, удаленный мониторинг и автоматизированные системы управления для обеспечения качества и долговечности хранимой продукции.

SUMMARY. This article explores the role of advanced sensors in monitoring and regulating essential environmental parameters such as temperature, humidity, carbon dioxide, oxygen, and ethylene levels. By integrating IoT technology, these sensors enable real-time data collection, remote monitoring, and automated control systems to ensure the quality and longevity of stored produce.

BERNULLI SONLARINING BA’ZI TADBIQLARI

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Tayanch so‘zlar: Bernulli sonlari**,** darajali qatorlar, yaqinlashish radiusi, chekli yig‘indi.

Ключевые слова: числа Бернулли, степенные ряды, радиус схадимости, конечная сумма.

Key words: Bernoulli numbers, power series, radius of convergence, terminating sum.

Bernulli sonlari matematikada juda muhim va qiziqarli konsepsiyalardan biridir. Ushbu sonlar asosan kombinatorika, ehtimollar nazariyasi, differensial tenglamalar va analiz kabi sohalarda keng qo‘llaniladi. Bernulli sonlari, shuningdek, turli xil matematik qatorlarni hisoblashda, masalan, trigonometrik funksiyalarni yoyishda yoki zeta funksiyalarini o‘rganishda muhim rol o‘ynaydi. Bu sonlar Jakob Bernulli sharafiga nomlangan bo‘lib, u ularni birinchi marta o‘rganib, ularning xususiyatlarini aniqlagan.

Bernulli sonlari butun sonlar va ratsional sonlar bilan chambarchas bog‘liq bo‘lib, ularning o‘ziga xos rekursiv formulalari va generator funksiyalari orqali aniqlanadi. Ular yordamida maxsus qatorlarni soddalashtirish, differensial va integral tenglamalarni yechishda aniq formulalar hosil qilish, shuningdek, sonlar nazariyasidagi qiyin muammolarni osonlashtirish mumkin. Aynan shuning uchun Bernulli sonlari nafaqat nazariy matematika, balki muhandislik, fizika va hatto moliyaviy hisob-kitoblarda ham muhim rol o‘ynaydi.

Kompleks sonlar tabiiy fanlarning turli sohalarida keng qo‘llaniladi. Ular yordamida mexanika va fizikaning ko‘pgina masalalari o‘z yechimini topgan. Shuningdek, bu sonlar algebraik va geometrik tatbiqlarga ega. Ushbu maqolada kompleks o‘zgaruvchili darajali qatorlar nisbati yordamida natural sonlar qatorining natural darajali yig‘indilarini hisoblash uchun umumiy formula keltirilib chiqarilgan.

Dastlab ikkita darajali qator nisbatini darajali qatorga yoyishni ko‘rib chiqamiz. Aytaylik, bizga ikkita

$$f_1(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \tag{1}$$

va

$$f_2(z) = b_0 + b_1 z + \ldots + b_n z^n + \ldots \tag{2}$$

darajali qatorlar berilgan bo‘lib, bu darajali qatorning yaqinlashish radiuslari mos ravishda musbat r va ρ bo‘lsin. Bunda (2) qator uchun $f_2(0) = b_0 \neq 0$ σ orqali r va ρ larnig kichigini belgilaymiz: $\sigma = \min(r, \rho)$. U holda $|z| < \sigma$ doirada ikkala qator ham yaqinlashadi. Agar bu doirada $f_2(z)$ funksiyaning nollari bo‘lsa, radiusi σ dan kichik bo‘lgan shunday doira olamizki, yangi doirada $f_2(z)$ nolga aylanmaydi ($b_0 \neq 0$). Bu doiraning radiusi R bo‘lsin. $|z| < R$ doira ichida

$$f(z)=\frac{a_0+a_1z+...+a_nz^n+...}{b_0+b_1z+...+b_nz^n+...} \tag{3}$$

nisbat mavjud bo‘ladi. Ushbu

$$f(z)=c_0+c_1z+...+c_nz^n+... \tag{4}$$

qatorni yuqorida berilgan qatorlarning bo‘linmasi deyishimiz uchun quyidagi ayniyatlar bajarilishi kerak:

$$\left[c_0+c_1z+...+c_nz^n+...\right]\cdot\left[b_0+b_1z+...+b_nz^n+...\right]=$$
$$=a_0+a_1z+...+a_nz^n+....$$

Darajali qatorlar $|z|<R$ doirada absolyut yaqinlashuvchi bo‘lgani uchun hadlab ko‘paytirishimiz mumkin.

$$\begin{gathered}c_0b_0+(c_0b_1+c_1b_0)z+(c_0b_2+c_1b_1+c_2b_0)z^2+\\+(c_0b_n+c_1b_{n-1}+...+c_nb_0)z^n+...=\\=a_0+a_1z+...+a_nz^n+...\end{gathered} \tag{5}$$

Bu tenglik chap va o‘ng tomonidagi qatorlarning mos koeffitsiyentlari o‘zaro teng bo‘ladi:

$$\left.\begin{array}{l}c_0b_0=a_0\\ c_0b_1+c_1b_0=a_1\\ c_0b_2+c_1b_1+c_2b_0=a_2\\ \dots\dots\dots\dots\dots\dots\\ c_0b_n+c_1b_{n-1}+c_2b_{n-2}+.....+c_nb_0=a_n\\ \dots\dots\dots\dots\dots\dots\dots\dots\end{array}\right\} \tag{6}$$

Bu sistema

$$c_0,c_1,c_2,\dots,c_n,\dots \tag{7}$$

noma’lum koeffitsiyentlarga nisbatan chiziqli tenglamalarning cheksiz sistemasidir. Shartga ko‘ra, $b_0\neq 0$ bo‘lgani uchun bu sistema noma’lumlarga nisbatan ketma-ket yechiladi.

Ikkinchi tomondan, c_n koeffitsiyentni $a_0,a_1,a_2,\dots,a_n$ va $b_0,b_1,b_2,\dots,b_n$ koeffitsiyentlar orqali determinantlar yordamida ifodalash mumkin. Buning uchun yuqoridagi sistemadan dastlabki $n+1$ ta tenglamani olib, unga Kramer qoidasini qo’llaymiz:

$$c_n=\frac{1}{b_0^{n+1}}\begin{vmatrix}b_0&0&0&\dots&a_0\\ b_1&b_0&0&\dots&a_1\\ b_2&b_1&b_0&\dots&a_2\\ \dots&\dots&\dots&\dots&\dots\\ b_n&b_{n-1}&b_{n-2}&\dots&a_n\end{vmatrix}, \tag{8}$$

Yuqoridagi mulohazalarni quyidagi misolga tatbiq qilamiz.

1-misol. $F(z)=\dfrac{z}{e^z-1}$ funksiyani darajali qatorga yoyamiz (qarang **[1]**). Bu funksiya butun kompleks $\mathbb{C}$ tekislikdan e^z-1 funksiyaning nollarini chiqarib tashlangan sohada, ya’ni bu funksiya $\mathbb{C}\setminus\{0,\pm 2\pi i,\pm 4\pi i,\dots\}$ sohada golomorf. Agar biz ushbu

$$e^z-1=\frac{z}{1}+\frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+... \tag{9}$$

yoyilmasida $F(z)$ ning ifodasiga qo’yib, z ga qisqartirib yuborsak, $F(z)$ funksiya quyidagi ko‘rinishni oladi:

$$F(z)=\frac{1}{1+\frac{z}{2!}+...+\frac{z^n}{(n+1)!}+...}.$$

Bu nisbatning maxraji $|z|<2\pi$ doirada nolga aylanmaydi. Demak, bu doirada $F(z)$ funksiyani darajali qatorga yoyishimiz mumkin. Agar bu nisbatning surati faqat bir bo‘lgani va mahraji yoyilma bo‘lganini e’tiborga olsak,

$$\begin{gathered}1=a_0+a_1z+a_2z^2+...+a_nz^n+\\+...\Rightarrow a_0=1,a_n=0\ (n=1,2,3,...).\end{gathered}$$

$$1+\frac{z}{2!}+...+\frac{z^n}{(n+1)!}+...=b_0+b_1z+...+b_nz^n+...\Rightarrow$$

$$\Rightarrow b_n=\frac{1}{(n+1)!},(n=0,1,2,...)$$

bo‘ladi. (6) tenglamalar sistemasining birinchi tenglamasida $c_0\cdot 1=1$ ya’ni $c_0=1$ ekanligi kelib chiqadi. Bu boshlang‘ich koeffitsiyent yordamida izlanayotgan qatorning koeffitsiyentlarini (6) tenglamalar sisemasining $n+1$ - tenglamasi yordamida

$$c_0\frac{1}{(n+1)!}+c_1\frac{1}{n!}+...+c_{n-1}\frac{1}{2!}+c_n=0,\ n=1,2,3,... \tag{10}$$

to‘la aniqlashimiz mumkin. Oxirgi tenglamadan c_n koeffitsiyent undan oldingi koeffisentlar yordamida aniqlanadi. Ikkinchi tomondan c_n koeffitsiyentni (9) formuladan ham topishimiz mumkin:

$$c_n=\begin{vmatrix}1&0&0&\dots&1\\ \frac{1}{2!}&1&0&\dots&0\\ \frac{1}{3!}&\frac{1}{2!}&1&\dots&0\\ \dots&\dots&\dots&\dots&0\\ \frac{1}{(n+1)!}&\frac{1}{n!}&\frac{1}{(n-1)!}&\dots&0\end{vmatrix}=$$

$$=(-1)^n\begin{vmatrix}\frac{1}{2!}&1&0&\dots&0\\ \frac{1}{3!}&\frac{1}{2!}&1&\dots&0\\ \dots&\dots&\dots&\dots&\dots\\ \dots&\dots&\dots&\dots&\dots\\ \frac{1}{(n+1)!}&\frac{1}{n!}&\frac{1}{(n-1)!}&\dots&\frac{1}{2!}\end{vmatrix}(n=1,2,3,\dots).$$

1-ta’rif. Ushbu $c_nn!$ sonlarga Bernulli sonlari deyiladi (Yakob Bernulli 1654-1705) va B_n kabi belgilanadi:

$$B_n=c_nn!.$$

Bu sonlar ko‘rinishidan juda oddiy sonlar bo‘lsa-da, matematikada ular yordamida juda ko‘p masalalarni yechishimiz mumkin. Bunga biz ushbu maqolaning oxirida batafsil to‘xtalamiz.

Bernulli sonlarini hisoblash uchun quyidagicha amallarni bajaramiz:

$$B_0 = c_0 \cdot 0! = 1,$$

$$B_n = c_n \cdot n! = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} \end{vmatrix} (n = 1,2,3\dots).$$

Boshqacha qilib aytganda, (10) tenglamadan foydalansak, bu sonlar ketma-ketligi uchun osongina quyidagi tenglikka ega bo‘lamiz:

$$B_0 \cdot \frac{1}{0!\,(n+1)!} + B_1 \cdot \frac{1}{1!\,n!} + \dots + B_n \cdot \frac{1}{n!\,1!} = 0 \ (n = 1,2,3\dots).$$

Oxirgi tenglikni $(n+1)!$ ga ko‘paytirsak hamda binomial koeffitsiyent uchun

$$\frac{(n+1)!}{k!(n+1-k)!} = C_{n+1}^k$$

tenglik o‘rinli ekanligini e’tiborga olsak, ushbu

$$B_0 C_{n+1}^0 + B_1 C_{n+1}^1 + B_2 C_{n+1}^2 + \dots +$$
$$+ B_n C_{n+1}^n = 0 \ (n = 1,2,3\dots)$$

munosabat kelib chiqadi. Bu formulani quyidagicha tasvirlash mumkin:

$$(1+B)^{n+1} - B^{n+1} = 0 \qquad (11)$$

Bu formulani shu ma’nosida tushunish lozimki, bunda darajali qavs ochilganda binomial koeffitsiyentlar o‘z o‘rnida, darajalar esa indekslar bilan almashtiriladi. $B_0 = 1$ deb qarab, qolgan Bernulli sonlarini (11) formula yordamida hisoblab chiqamiz:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},$$
$$B_5 = 0,\ B_6 = \frac{1}{42},\dots$$

Demak, misolda berilgan funksiya ushbu

$$\frac{z}{e^z - 1} = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots = \qquad (12)$$
$$= B_0 + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \dots + \frac{B_n}{n!} z^n + \dots$$

ko‘rinishdagi darajali qatorga yoyiladi.

Agar (12) yoyilmada z ning o’rniga $-z$ ni qo’yib, hosil bo’lgan yoyilmani (12) yoyilmadan ayirsak, Bernulli sonlarining birdan katta toq nomerli hadlari nolga teng ekanligi kelib chiqadi:

$$\forall k \in \mathbb{N} \text{ da } B_{2k+1} = 0.$$

Bundan ushbu

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k},\ (|z| < 2\pi) \qquad (13)$$

yoyilma o‘rinli bo‘ladi.

Endi (13) yoyilmaning tatbiqini keltiramiz. Ma’lumki, quyidagi

$$1^p + 2^p + 3^p + \dots + n^p = \sum_{j=1}^{n} j^p, (p \in \mathbb{N}) \qquad (14)$$

yig‘indi amaliy masalalarda tez-tez uchrab turadi. Agar $p = 1,2,3$ bo‘lsa bu yig‘indini hisoblash ancha oson. Ammo $p \geq 4$ bo‘lgan hollarda yig‘indini hisoblash qiyinchilik tug‘diradi.

Biz quyida (14) yig‘indi uchun umumiy formulani keltirib chiqaramiz.

1-teorema. *Ixtiyoriy* $N,\ p \in \mathbb{N}$ *soni uchun ushbu*

$$1^p + 2^p + \dots + (N-1)^p = \sum_{j=0}^{N-1} j^p = \sum_{k=0}^{p} C_{p+1}^{k+1} B_{p-k} \frac{N^{k+1}}{p+1} \qquad (15)$$

tenglik o’rinli bo‘ladi. Bu yerda $B_n, (n = 0,1,2,\dots)$- *Bernulli sonlari,* $C_n^k = \frac{n!}{k!\,(n-k)!}$.

Isbot. Dastlab quyidagi

$$1 + e^z + e^{2z} + \dots + e^{(N-1)z} = \sum_{j=0}^{N-1} e^{jz}$$

geometrik progressiya yig‘indisidan foydalanamiz va quyidagicha belgilashlar kiritamiz:

$$\sum_{j=1}^{N-1} e^{jz} = \frac{e^{Nz} - 1}{e^z - 1} = \frac{e^{Nz} - 1}{z} \cdot \frac{z}{e^z - 1} = f(z) \cdot g(z), \qquad (16)$$

$$f(z) = \frac{e^{Nz} - 1}{z},\ g(z) = \frac{z}{e^z - 1}.$$

Ma’lumki, ikkita funksiyaning ko‘paytmasidan n-tartibli hosila olishning Leybnits qoidasiga ko‘ra, ya’ni agar $u = \varphi(z),\ v = \psi(z)$ funksiyalar $n-$ tartibli differensiallanuvchi funksiyalar bo‘lsa, u holda:

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}$$

formula o‘rinli. Bu yerda $u^{(0)} = u,\ v^{(0)} = v$.

(16) formuladan p- tartibli hosila olib, $z = 0$ dagi qiymati topilsa,

$$\sum_{j=0}^{N-1} j^p = \left(\frac{d}{dz}\right)^p \left(f(z)g(z)\right)(0) = \qquad (17)$$
$$= \sum_{k=0}^{p} C_p^k f^{(k)}(0) g^{(p-k)}(0)$$

bo‘ladi.

$f(z) = \frac{e^{Nz} - 1}{z}$ funksiyaning $k-$ tartibli hosilasini topish uchun uni $z = 0$ nuqta atrofida z ning darajalari bo‘yicha darajali qatorga yoyamiz:

$$f(z) = \frac{1}{z}\left(1 + \frac{Nz}{1!} + \frac{(Nz)^2}{2!} + \dots + \frac{(Nz)^n}{n!} - 1\right) =$$
$$= \frac{N}{1!} + \frac{N^2 z}{2!} + \frac{N^3 z^2}{3!} + \dots + \frac{N^n z^{n-1}}{n!} + \frac{N^{n+1} z^n}{(n+1)!} + \dots$$

bundan

$$f^{(k)}(0)=\frac{N^{k+1}}{k+1}$$

ekanligi kelib chiqadi. Shuningdek $g(z)=\dfrac{z}{e^{z}-1}$ funksiyaning $(p-k)$- tartibli hosilasini topish uchun ushbu

$$g(z)=g(0)+\frac{g'(0)}{1!}z+\frac{g''\ \ 0}{2!}z^{2}+...+\frac{g^{(0)}(0)}{n!}z^{n}+...$$

tenglikdan foydalanamiz. Buni va (12) formulani hisobga olsak, ushbu

$$g^{\ p-k}\ (0)=B_{p-k}$$

tenglik kelib chiqadi. Bunga asosan (16) formula quyidagi koʻrinishni oladi:

$$\sum_{k=0}^{N-1} j^{p}=\sum_{k=0}^{p} C_{p}^{k}B_{p-k}\,\frac{N^{k+1}}{k+1} \tag{18}$$

ekanligi kelib chiqadi. Agar

$$\frac{C_{p}^{k}}{k+1}=\frac{C_{p+1}^{k+1}}{p+1}$$

ayniyatni e'tiborga olsak, (18) formuladan

$$\sum_{k=0}^{N-1} j^{p}=\sum_{k=0}^{p} C_{p}^{k}B_{p-k}\,\frac{N^{k+1}}{k+1}=$$

$$=\sum_{k=0}^{p} C_{p+1}^{k+1}\,B_{p-k}\,\frac{N^{k+1}}{p+1}$$

ekanligi, ya'ni (15) formula oʻrinli ekanligi kelib chiqdi. Teorema isbotlandi.

Topilgan (15) formuladan kelib chiqadigan bir qancha natijalarni koʻrib chiqamiz. $p=4$ da (15) formula quyidagi koʻrinishda boʻladi:

$$1^{4}+2^{4}+3^{4}+\ldots+(n-1)^{4}=$$

$$=\sum_{k=0}^{n-1} j^{4}=\sum_{k=0}^{4} C_{5}^{k+1}B_{4-k}\,\frac{n^{k+1}}{5}=$$

$$=\frac{1}{5}\ \ C_{5}^{1}B_{4}n+C_{5}^{2}B_{3}n^{2}+C_{5}^{3}B_{2}n^{3}+C_{5}^{4}B_{1}n^{4}+C_{5}^{5}B_{0}n^{5}\ \ =$$

$$=\frac{n^{5}}{5}-\frac{n^{4}}{2}+\frac{n^{3}}{3}-\frac{n}{30}$$

ya'ni ushbu

$$1^{4}+2^{4}+3^{4}+\ldots+n^{4}=$$

$$\frac{1}{5}n^{5}+\frac{1}{2}n^{4}+\frac{1}{3}n^{3}-\frac{1}{30}n$$

formula o'rinli bo'ladi.

Bu topilgan (15) formuladan kelib chiqadigan bir qancha natijalarni koʻrib chiqamiz.

$p=1$ da (15) formula quyidagi koʻrinishda boʻladi:

$$1+2+3+\ldots+\ \ n-1\ \ =\sum_{k=0}^{p} C_{p+1}^{k+1}\,B_{p-k}\,\frac{N^{k+1}}{p+1}=$$

$$=\sum_{k=0}^{1} C_{2}^{k+1}B_{1-k}\cdot\frac{n^{k+1}}{1+1}=\frac{1}{2}\ \ C_{2}^{1}B_{1}n+C_{2}^{2}B_{0}n^{2}\ \ =$$

$$=\frac{1}{2}\left(2\cdot(-\frac{1}{2})\cdot n+n^{2}\right)=\frac{1}{2}\ \ n^{2}-n\ \ =\frac{n\ \ n-1}{2}$$

Bundan esa

$$1+2+3+\ldots+n=\frac{n\ \ n+1}{2}$$

$p=2$ boʻlsa

$$\sum_{k=0}^{n-1} j^{2}=\sum_{k=0}^{2} C_{3}^{k+1}B_{2-k}\,\frac{n^{k+1}}{3}=\frac{1}{3}\ \ C_{3}^{1}B_{2}n+C_{3}^{2}B_{1}n^{2}+C_{3}^{3}B_{0}n^{3}\ \ =$$

$$=\frac{1}{3}\left(3\cdot\frac{1}{6}n+3\cdot\left(-\frac{1}{2}\right)n^{2}+n^{3}\right)=\frac{1}{3}\left(\frac{n}{2}-\frac{3n^{2}}{2}+n^{3}\right)=$$

$$=\frac{1}{3}\cdot\frac{n}{2}\ \ 2n^{2}-3n+1\ \ =\frac{n\ \ n-1\ \ 2n-1}{6}$$

bundan

$$\sum_{k=0}^{n-1} j^{2}=1^{2}+2^{2}+3^{2}+...+(n-1)^{2}=\frac{n\ \ n-1\ \ 2n-1}{6}$$

$p=3$ boʻlsa

$$1^{3}+2^{3}+\ldots+\ \ n-1\ ^{3}=\sum_{k=0}^{n-1} j^{3}=\sum_{k=0}^{3} C_{4}^{k+1}B_{3-k}\,\frac{n^{k+1}}{4}=$$

$$=\frac{1}{4}\ \ C_{4}^{1}B_{3}n+C_{4}^{2}B_{2}n^{2}+C_{4}^{3}B_{1}n^{3}+C_{4}^{4}B_{0}n^{4}\ \ =$$

$$=\frac{1}{4}\left(4\cdot 0\cdot n+6\cdot\frac{1}{6}\cdot n^{2}+4\cdot\left(-\frac{1}{2}\right)\cdot n^{3}+n^{4}\right)=$$

$$=\frac{1}{4}\ \ n^{4}-2n^{3}+n^{2}\ \ =$$

$$=\left(\frac{n\ \ n-1}{2}\right)^{2}$$

ya'ni

$$1^{3}+2^{3}+\ldots+\ \ n-1\ ^{3}=\left(\frac{n\ \ n-1}{2}\right)^{2}$$

$p=4$ boʻlsa

$$1^{4}+2^{4}+3^{4}+\ldots+n^{4}=\sum_{k=0}^{n} j^{4}=$$

$$=\sum_{k=0}^{4} C_{4}^{k+1}B_{4-k}\,\frac{n+1\ ^{k+1}}{5}=$$

$$=\frac{1}{5}(C_{5}^{1}B_{4}\ \ n+1\ \ +C_{5}^{2}B_{3}\ \ n+1\ ^{2}+$$

$$+C_{5}^{3}B_{2}\ \ n+1\ ^{3}+C_{5}^{4}B_{1}\ \ n+1\ ^{4}+C_{5}^{5}B_{0}\ \ n+1\ ^{5})=$$

$$=\frac{1}{5}(5\cdot\left(-\frac{1}{30}\right)\ n+1\ +6\cdot 0\cdot\ n+1\ ^{2}+$$

$$+10\cdot\frac{1}{6}\cdot\ \ n+1\ ^{3}+5\cdot\left(-\frac{1}{2}\right)\ n+1\ ^{4}+\ \ n+1\ ^{5})=$$

$$=-\frac{1}{30}\cdot\ \ n+1\ +\frac{1}{3}\ \ n+1\ ^{3}-\frac{1}{2}\ \ n+1\ ^{4}+\frac{1}{5}\ \ n+1\ ^{5}=$$

$$=\frac{1}{5}n^{5}+\frac{1}{2}n^{4}+\frac{1}{3}n^{3}-\frac{1}{30}n$$

Demak,

$$\sum_{j=1}^{n} j^{4}=1^{4}+2^{4}+3^{4}+\ldots n^{4}=\frac{1}{5}n^{5}+\frac{1}{2}n^{4}+\frac{1}{3}n^{3}-\frac{1}{30}n$$

$p=5$ boʻlgan holda

$$\sum_{k=0}^{n} j^{5}=\sum_{k=0}^{5} C_{5}^{k+1}B_{5-k}\,\frac{n+1\ ^{k+1}}{6}=$$

$$=\frac{1}{6}(C_{6}^{1}B_{5}\ \ n+1\ \ +C_{6}^{2}B_{4}\ \ n+1\ ^{2}+C_{6}^{3}B_{3}\ \ n+1\ ^{3}+$$

$+C_6^4B_2(n+1)^4 + C_6^5B_1(n+1)^5 + C_6^6B_0(n+1)^6) =$

$= \frac{1}{6}(6\cdot 0\cdot (n+1) + 15\cdot\left(-\frac{1}{30}\right)(n+1)^2 + 20\cdot 0\cdot (n+1)^3 +$

$+15\cdot\frac{1}{6}\cdot (n+1)^4 + 6\cdot\left(-\frac{1}{2}\right)(n+1)^5 + (n+1)^6)$

$= -\frac{1}{12}\cdot (n+1)^2 + \frac{5}{12}(n+1)^4 - \frac{1}{2}(n+1)^5 + \frac{1}{6}(n+1)^6 =$

$= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$

ya'ni

$1^5 + 2^5 + 3^5 + \ldots + n^5 = \sum_{j=1}^{n} j^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$

$p = 6$ bo'lsa

$$\sum_{k=0}^{n} j^6 = \sum_{k=0}^{6} C_7^{k+1}B_{6-k}\frac{(n+1)^{k+1}}{7} =$$

$= \frac{1}{7}(C_7^1B_6(n+1) + C_7^2B_5(n+1)^2 + C_7^3B_4(n+1)^3 +$

$+C_7^4B_3(n+1)^4 + C_7^5B_2(n+1)^5 + C_7^6B_1(n+1)^6 + C_7^7B_0(n+1)^7) =$

$= \frac{1}{7}(7\cdot\frac{1}{42}\cdot (n+1) + 21\cdot 0\cdot (n+1)^2 + 35\cdot\left(-\frac{1}{30}\right)(n+1)^3 +$

$+35\cdot 0\cdot (n+1)^4 + 21\cdot\frac{1}{6}\cdot (n+1)^5 +$

$+7\cdot\left(-\frac{1}{2}\right)(n+1)^6 + (n+1)^7) =$

$= \frac{1}{42}\cdot (n+1) - \frac{1}{6}(n+1)^3 + \frac{1}{2}(n+1)^5 - \frac{1}{2}(n+1)^6 + \frac{1}{7}(n+1)^7 =$

$= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

Demak,

$$1^6 + 2^6 + 3^6 + \ldots + n^6 = \sum_{j=1}^{n} j^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^4 + \frac{1}{42}n$$

Xuddi shunga o'xshash $p = 10$ bo'lsa,

$$\sum_{j=1}^{n} j^{10} = 1^{10} + 2^{10} + 3^{10} + \ldots + n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

ekanligi kelib chiqadi.

Umuman olgandan Bernulli sonlari bilan bog'liq ravishda xuddi shunga o'xshash $\sum_{j=0}^{n} j^p$ yig'indini p parametr ma'lum bo'lgan aniq n ga nisbatan ifodasini topish mumkinligi ko'rib chiqildi.

Adabiyotlar

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REZYUME. Ushbu maqolada kompleks o'zgaruvchili darajali qatorlar nisbatining ba'zi tatbiqlari keltirilgan bo'lib, bunda Bernulli sonlari yordamida $1^p + 2^p + 3^p + ... + n^p = \sum_{j=1}^{n} j^p$ yig'indini hisoblash usuli ixtiyoriy natural p uchun keltirilib chiqarilgan. Maqola bakalavriatning matematik yo'nalishi katta kurs talabalari va magistratura talabalari uchun mo'ljallangan.

РЕЗЮМЕ. В данной работе рассматривается некоторые применение частное степенных рядов комплексных переменных, где основную роль играет, так называемое, числа Бернулли. В частности, с помощью этих чисел, доказано равенство $1^p + 2^p + 3^p + ... + n^p = \sum_{j=1}^{n} j^p$ для любых натуральных p. Статья может быть интересным для студентов старших курсов математического направления бакалавриата и студентов магистратуры.

SUMMARY. In this article, there are considered some applications of complex variable power series, in which with the help of Bernoulli's numbers, the method of calculating the sum $1^p + 2^p + 3^p + ... + n^p = \sum_{j=1}^{n} j^p$ for an arbitrary natural number p is given. The article may be of interest to senior undergraduate students majoring in mathematics and master's degree students.